PROBABILITY AND RANDOM VARIABLES

Definitions:

Deterministic experiments:-

There are experiments which always produce the same result.

Random experiments:-

The experiments which do not produce the same result.

Trial & event:-

The performance of a random experiment is called a trial & the outcome is called an event.

E.g: Throwing of a coin is a trial & getting H or T is an event.

Sample space:-

The totality of the possible outcomes of a random experiment is called the sample space of the experiment.

Equally likely events:-

The possibilities or events are said to be equally likely when we have no reason to expect any one rather than the other.

E.g: In tossing an unbiased coin, the head or tail are equally likely.

Mutually exclusive events:-

If A & B are mutually exclusive, then it is not possible for both events to occur on the same trial.

Exhaustive events:-

Events are said to be exhaustive when they include all possibilities.

Favourable events:-

The trails which entail the happening of an event are said to be favourable to the event.

Probability:- Chance of happening.

$$P(A) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}}.$$

Permutation :-

Selection & arrangement of factors.

$$np_r = \frac{n!}{(n-r)!}$$

Permutations with repetitions :-

Let $p(n: n_1, n_2, \ldots, n_r)$ denotes the number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike, $p(n: n_1, n_2, \ldots, n_r) = \frac{n!}{n_1!\, n_2! \ldots n_r!}$.

Eg: The number of permutations of the word 'RADAR' is $\frac{5!}{2!\, 2!} = 30$.

Combination:-

Combinations means selection of factors.

$$nc_r = \frac{np_r}{r!} = \frac{n!}{(n-r)!\, r!}$$

Note:

$nc_n = nc_0 = 1$

$nc_r = nc_{n-r}$

Axioms of Probability:

If S is the sample space & E is any event in a random experiment,

Axiom 1: $0 \le P(E) \le 1$.

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Problems:

① If 3 balls are randomly drawn from a bowl containing 6 white & 5 black balls, what is the probability that one of the drawn ball is white & the other two black?

Sol: P[One of the drawn ball is white & the other two are black] $= P(A) = \dfrac{n(A)}{n(S)}$

$$= \frac{6C_1 \times 5C_2}{11C_3} = \frac{60}{165} = \frac{4}{11}$$

② A lot of integrated circuit chips consists of 10 good, 4 with minor defects & 2 with major defects. Two chips are randomly chosen from the lot. What is the probability that atleast one chip is good?

Sol: P[atleast one chip is good] $= P(A) = \dfrac{n(A)}{n(S)}$

$$= \frac{(10C_1 \times 6C_1) + 10C_2}{16C_2} = \frac{60+45}{120} = \frac{105}{120} = \frac{7}{8}$$

③ 4 persons are chosen at random from a group containing 3 men, 2 women & 4 children. Show that the chance that exactly 2 of them will be children is $\frac{10}{21}$.

Sol: P[exactly 2 of them will be children] $= P(A) = \dfrac{n(A)}{n(S)}$

$$= \frac{4C_2 \times 5C_2}{9C_4} = \frac{60}{126} = \frac{10}{21}$$

(4) From a group of 5 first year, 4 second year & 4 third year students, 3 students are selected at random. Find the probability that they are first year or third year students.

Sol:

P[they are first year or third year students] = $P(A) = \frac{n(A)}{n(S)}$

$$= \frac{5C_3 + 4C_3}{13C_3} = \frac{10+4}{286} = \frac{14}{286} = \frac{7}{143}$$

(5) A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails & 1 head.

Sol: Given $P(H) = \frac{2}{3}$ & $P(T) = \frac{1}{3}$.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$P(HTT) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$; $P(THT) = \frac{2}{27}$; $P(TTH) = \frac{2}{27}$

$\therefore$ P[getting 2 tails & 1 head] $= \frac{2}{27} + \frac{2}{27} + \frac{2}{27} = \frac{6}{27}$

(6) One card is drawn from a deck of 52 cards. What is the probability of the card being either red or a king?

Sol: WKT $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

P[the card being either red or a king] $= \frac{26C_1 + 4C_1 - 2C_1}{52C_1}$

$$= \frac{26+4-2}{52} = \frac{28}{52} = \frac{7}{13}$$

(7) If A & B are events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ & $P(A \cap B) = \frac{1}{4}$, find $P(A^c \cap B^c)$.

Sol:

$P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - \left[\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right] = 1 - \frac{5}{8} = \frac{3}{8}$

Let events A & B be independent with $P(A)=0.5$ & $P(B)=0.8$. Find the Probability that neither of the events A nor B occurs.

Sol:

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1-P(A\cup B)$$

$$= 1-[P(A)+P(B)-P(A\cap B)]$$

$$= 1-[P(A)+P(B)-P(A)\cdot P(B)] \quad [\because A \text{ \& } B \text{ are independent}]$$

$$= 1-[0.5+0.8-(0.5\times 0.8)]$$

$$= 1-[1.3-0.4] = 1-0.9 = 0.1$$

(9) Event A & B are such that $P(A+B)=\frac{3}{4}$, $P(AB)=\frac{1}{4}$ & $P(\bar{A})=\frac{2}{3}$ find $P(B)$.

Sol: $P(A)=1-P(\bar{A})=1-\frac{2}{3}=\frac{1}{3}$

$$P(A+B) = P(A)+P(B)-P(AB)$$

$$\frac{3}{4} = \frac{1}{3}+P(B)-\frac{1}{4}$$

$$\therefore P(B) = \frac{3}{4}-\frac{1}{3}+\frac{1}{4} = 1-\frac{1}{3} = \frac{2}{3}$$

(10) A total of 36 members of a club play Tennis, 28 play squash, & 18 play badminton. Furthermore, 22 of the members play both tennis & squash 12 play both tennis & badminton, 9 play both squash & badminton, & 4 play all the 3 sports. How many members of this club play atleast one of these sports?

Sol:

$$P[\text{play atleast one of these sports}] = P(T\cup S\cup B)$$

$$= P(T)+P(S)+P(B)-P(T\cap S)-P(S\cap B)-P(T\cap B)+P(T\cap S\cap B)$$

$$= \frac{36+28+18-22-9-12+4}{N} = \frac{43}{N}$$

Hence 43 members play atleast one of these sports.

(11) Out of $(2n+1)$ tickets consecutively numbered three are drawn at random. Find the probability that the numbers on them are in arithmetic progression.

Sol: $n(S) = (2n+1)C_3 = \frac{(2n+1)\,2n\,(2n-1)}{3!} = \frac{n(4n^2-1)}{3}$

$d = 1, 2, 3, \ldots, (n-1), n$ (Difference)

If $d=1$

$$\left.\begin{array}{l} 1, 2, 3 \\ 2, 3, 4 \\ \vdots \\ (2n-1)\,2n\,(2n+1) \end{array}\right\} \text{totally } (2n-1) \text{ cases}$$

If $d=2$

$$\left.\begin{array}{l} 1, 3, 5 \\ 2, 4, 6 \\ \vdots \\ (2n-3)(2n-1)(2n+1) \end{array}\right\} \text{totally } (2n-3) \text{ cases}$$

If $d=n-1$

$$\left.\begin{array}{l} 1, n, 2n-1 \\ 2, n+1, 2n \\ 3, n+2, 2n+1 \end{array}\right\} \text{totally } 3 \text{ cases}$$

If $d=n$ $\quad 1, n+1, 2n+1 \rightarrow$ totally 1 case

$\therefore n(A) = (2n-1) + (2n-3) + \cdots + 3 + 1 = \frac{n}{2}(1+(2n-1)) = \frac{n}{2} \times 2n = n^2$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{n^2}{n(4n^2-1)} \times 3 = \frac{3n}{4n^2-1}$

(12) A can hit a target in 4 out of 5 shots & B can hit the target in 3 out of 4 shots. Find the probability that (i) the target being hit when both try. (ii) the target being hit by exactly one person.

Sol: $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) \cdot P(B)$

$= \frac{4}{5} + \frac{3}{4} - \left(\frac{4}{5} \cdot \frac{3}{4}\right) = \frac{19}{20}$

(ii) P[The target being hit by exactly one person]

$= P[(A \cap \bar{B}) \cup (B \cap \bar{A})] = P[(A \cap \bar{B}) + (B \cap \bar{A})]$

$= P(A)\,P(\bar{B}) + P(B)\,P(\bar{A})$

$= P(A)(1 - P(B)) + P(B)(1 - P(A))$

$= \frac{4}{5}\left(1 - \frac{3}{4}\right) + \frac{3}{4}\left(1 - \frac{4}{5}\right)$

$$=\left(\frac{4}{5}\times\frac{1}{4}\right)+\left(\frac{3}{4}\times\frac{1}{5}\right)$$

$$=\frac{1}{5}+\frac{3}{20}=\frac{7}{20}$$

Marginal Probability:

A probability of only one event that takes place is called a marginal probability.

Joint Probability:

The probability of occurrence of both events A & B together, denoted by $P(A\cap B)$, is known as joint probability of A & B.

Conditional Probability:

The conditional probability of A given B is

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$ if $P(B)\neq 0$ & it is undefined otherwise.

Problems:

1. When 2 dice are thrown (or a die is thrown twice). Let A be the event that the sum of the points on the faces is odd & B is the event that atleast one number is 2. Find the probabilities of the following:
(1) A (2) B (3) $\bar{A}$ (4) $\bar{B}$ (5) $A\cap B$ (6) $A\cup B$ (7) $\bar{A}\cap B$ (8) $A\cap\bar{B}$ (9) $\bar{A}\cap\bar{B}$ (10) $\bar{A}\cup B$ (11) $A\cup\bar{B}$ (12) $\bar{A}\cup\bar{B}$ (13) A/B (14) B/A.

Sol: The sample space is

$$S=\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\ \vdots\\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{matrix}\right\}$$

$n(S)=36$

$A=\{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),$
$(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$n(A)=18$

$B=\{(1,2),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),$
$(6,2)\}$

$n(B)=11$

$A\cap B = \{(1,2),(2,1),(2,3),(2,5),(3,2),(5,2)\}$

$n(A\cap B) = 6$

① $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{18}{36} = \dfrac{1}{2}$ ② $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{11}{36}$

③ $P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$ ④ $P(\overline{B}) = 1 - P(B) = 1 - \frac{11}{36} = \frac{25}{36}$

⑤ $P(A\cap B) = \dfrac{n(A\cap B)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$

⑥ $P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{1}{2} + \frac{11}{36} - \frac{1}{6} = \frac{23}{36}$

⑦ $P(\overline{A}\cap B) = P(B) - P(A\cap B) = \frac{11}{36} - \frac{1}{6} = \frac{5}{36}$

⑧ $P(A\cap \overline{B}) = P(A) - P(A\cap B) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

⑨ $P(\overline{A}\cap \overline{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

⑩ $P(\overline{A}\cup B) = P(\overline{A}) + P(B) - P(\overline{A}\cap B) = \frac{1}{2} + \frac{11}{36} - \frac{5}{36} = \frac{2}{3}$

⑪ $P(A\cup \overline{B}) = P(A) + P(\overline{B}) - P(A\cap \overline{B}) = \frac{1}{2} + \frac{25}{36} - \frac{1}{3} = \frac{31}{36}$

⑫ $P(\overline{A}\cup \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A}\cap \overline{B}) = \frac{1}{2} + \frac{25}{36} - \frac{13}{36} = \frac{5}{6}$

⑬ $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{1/6}{11/36} = \dfrac{6}{11}$

⑭ $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$

Baye's theorem: (Theorem of probability of cases)

Let $B_1, B_2, \dots, B_n$ be an exhaustive & mutually exclusive random experiments & A be an event related to that B_i then

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A|B_i)}$$

Proof: According to conditional probability

$$P(B_i|A) = \frac{P(B_i\cap A)}{P(A)} \quad \text{—①}$$

Using multiplication rule of probability

$$P(B_i \cap A) = P(B_i)\,P(A|B_i) \quad \text{--- (2)}$$

Using total probability theorem

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A|B_i) \quad \text{--- (3)}$$

$$\therefore (1) \Rightarrow P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)} \quad \text{by (2) \& (3).}$$

Problems:

(1) The contents of urns I, II, III are as follows:

Urns \ Balls	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One urn is chosen at random & 2 balls are drawn. They happen to be white & red. What is the probability that they come from urns I, II & III?

Sol: Let B_1, B_2, B_3 denote the events that the urns I, II, III are chosen respectively & let A be the event that the 2 balls taken from the selected urn are white & red.

Then $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$.

$$P(A|B_1) = \frac{1C_1 \times 3C_1}{6C_2} = \frac{1}{5}$$

$$P(A|B_2) = \frac{2C_1 \times 1C_1}{4C_2} = \frac{1}{3}$$

$$P(A|B_3) = \frac{4C_1 \times 3C_1}{12C_2} = \frac{2}{11}$$

Baye's theorem: $P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\left(\frac{1}{3}\times\frac{1}{5}\right) + \left(\frac{1}{3}\times\frac{1}{3}\right) + \left(\frac{1}{3}\times\frac{2}{11}\right)} = \frac{\frac{1}{15}}{\frac{1}{15} + \frac{1}{9} + \frac{2}{33}} = \frac{33}{118}$$

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{15} + \frac{1}{9} + \frac{2}{33}} = \frac{55}{118}$$

$$P(B_3|A) = 1 - P(B_1|A) - P(B_2|A) = 1 - \frac{33}{118} - \frac{55}{118} = \frac{15}{59}$$

---

(2) Companies B_1, B_2 & B_3 produce 30%, 45% & 25% of the cars respectively. It is known that 2%, 3% & 2% of these cars produced from are defective. (i) What is the probability that a car purchased is defective? (ii) If a car purchased is found to be defective, what is the probability that this car is produced by company B_1?

Sol: Let x be the event that the car purchased is defective.

$P(B_1) = 30\% = \frac{30}{100} = 0.3$

$P(x/B_1) = 2\% = \frac{2}{100} = 0.02$

$P(B_2) = 45\% = \frac{45}{100} = 0.45$

$P(x/B_2) = 3\% = \frac{3}{100} = 0.03$

$P(B_3) = 25\% = \frac{25}{100} = 0.25$

$P(x/B_3) = 2\% = \frac{2}{100} = 0.02$

(i) $P(x) = P(B_1)P(x/B_1) + P(B_2)P(x/B_2) + P(B_3)P(x/B_3)$

$= (0.3 \times 0.02) + (0.45 \times 0.03) + (0.25 \times 0.02)$

$= 0.0245$

(ii) $P(B_1/x) = \frac{P(B_1)P(x/B_1)}{P(x)} = \frac{(0.3 \times 0.02)}{0.0245} = \frac{12}{49}$

---

3) A given lot of IC chips contains 2% defective chips. Each is tested before delivery. The tester itself is not totally reliable. Probability of tester says the chip is good when it is really good is 0.95 & the probability of tester says chip is defective when it is actually defective is 0.94. If a tested device is indicated to be defective, what is the probability that it is actually defective.

Sol:

E → Event of chip is actually good.

$\bar{E}$ → Event of chip is actually defective.

We know that $P(E) + P(\bar{E}) = 1$.

D → Event of tester says it is good.

$\bar{D}$ → Event of tester says it is defective.

Given: Lot of IC Chips contains 2% defective chips.

(ie) $P(\bar{E}) = 2\% = \frac{2}{100} = 0.02$

$P(E) = 1 - P(\bar{E}) = 1 - 0.02 = 0.98$

Given: Probl. of tester says the chip is good when it is really good is 0.95.

(ie) $P(D|E) = 0.95$

$P(\bar{D}|E) = 1 - P(D|E) = 1 - 0.95 = 0.05$

Given: Probl. of the tester says the chip is defective when it is actually defective is 0.94.

(ie) $P(\bar{D}|\bar{E}) = 0.94$

To find: The probl. of actually defective

(ie) $P(\bar{E}|\bar{D})$

By Baye's theorem,

$$P(\bar{E}|\bar{D}) = \frac{P(\bar{D}|\bar{E}) \cdot P(\bar{E})}{P(\bar{D}|\bar{E}) \cdot P(\bar{E}) + P(\bar{D}|E) \cdot P(E)}$$

$$= \frac{0.94 \times 0.02}{(0.94 \times 0.02) + (0.05 \times 0.98)} = 0.2773$$

(4) A bag contains 3 black & 4 white balls. Two balls are drawn at random one at a time without replacement.

(i) What is the probability that the second ball drawn is white?

(ii) What is the conditional probability that the first ball drawn is white if the second ball is known to be white?

Sol: Given: 3 black balls, 4 white balls

Total no. of balls = 3+4 = 7.

Let A → the first ball drawn is white.

B → the second ball drawn is white.

Second ball is white; it can happen in two mutually exclusive ways.

(1) First ball is white & second is white.

(2) First ball is black & second is white.

(i) $P(B) = P(1) + P(2) = \left(\frac{4}{7} \times \frac{3}{6}\right) + \left(\frac{3}{7} \times \frac{4}{6}\right)$

$$= \frac{12}{42} + \frac{12}{42} = \frac{24}{42} = \frac{4}{7}$$

(ii) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$A \cap B$ = first ball was white & second ball is also white.

$P(A \cap B) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

$\therefore P(A|B) = \frac{2/7}{4/7} = \frac{1}{2}$

(5) A consulting firm rents cars from 3 rental agencies in the following manner: 20% from agency D, 20% from agency E and 60% from agency F. If 10% cars from D, 12% of the cars from E & 4% of the cars from F have bad tyres. What is the probability that the firm will get a car with bad tyres? Find the firm will get a car with bad tyres? Find the prob. that a car with bad tyres is rented from agency F.

Sol: Let A be the event that the car has bad tyres.

Given: $P(D) = 20\% = 0.2$

$P(A|D) = 10\% = 0.1$

$P(E) = 20\% = 0.2$

$P(A|E) = 12\% = 0.12$

$P(F) = 60\% = 0.6$

$P(A|F) = 4\% = 0.04$

$$P(A) = P(D)\,P(A|D) + P(E)\,P(A|E) + P(F)\,P(A|F)$$

$$= (0.2 \times 0.1) + (0.2 \times 0.12) + (0.6 \times 0.04)$$

$$= 0.068$$

$$P(F|A) = \frac{P(A \cap F)}{P(A)} = \frac{P(F)\,P(A|F)}{P(A)}$$

$$= \frac{0.6 \times 0.04}{0.068} = \frac{6}{17}$$

Variance of X:

$$Var(X) = E[X^2] - [E(X)]^2$$

The quantity $\sqrt{Var(X)}$ is called the standard deviation of X.

Formulae:

① $\sum_{i=1}^{n} P(x_i) = 1$

② $F(x) = P(X \le x)$ (ie) E.g: $P(X \le 4) = F(4)$, $P(X \le 5) = F(5)$. $F(0) = P(0)$

$F(1) = P(0) + P(1)$

$F(2) = P(0) + P(1) + P(2) = F(1) + P(2)$

$F(3) = P(0) + P(1) + P(2) + P(3) = F(2) + P(3)$

③ $P(1) = F(1) - F(0)$

$P(2) = F(2) - F(1)$

$P(3) = F(3) - F(2)$

④ $\text{Mean} = E(X) = \sum_i x_i P(x_i)$

⑤ $E[X^2] = \sum_i x_i^2 P(x_i)$

⑥ $\text{Variance} = Var[X] = E[X^2] - [E(X)]^2$

⑦ $E[aX + b] = aE[X] + b$

⑧ $Var[aX \pm b] = a^2 Var(X)$

Problems:

① For the following probability distribution (i) Find the distribution func. of X. (ii) what is the smallest value of x for which $P(X \le x) > 0.5$.

Soln

x:	0	1	2
$P(x)$:	$1/4$	$2/4$	$1/4$

Sol: (i) The distribution func. of X is given by $F(x) = P(X \le x)$.

x	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = P(X = 0) = 1/4$
1	$F(1) = P(X \le 1) = P(X=0) + P(X=1) = 1/4 + 2/4 = 3/4$
2	$F(2) = P(X \le 2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$

(ii) The smallest value of x for which $P(X \le x) > 0.5$ is 1.

(2) Obtain the probability fun/. (or) probability distribution from the following distribution fun/..

x:	0	1	2	3
$F(x)$:	0.1	0.4	0.9	1

Sol: Probability fun/.:

x	$P(x)$
0	$F(0) = P(0) = 0.1$
1	$P(1) = F(1) - F(0) = 0.4 - 0.1 = 0.3$
2	$P(2) = F(2) - F(1) = 0.9 - 0.4 = 0.5$
3	$P(3) = F(3) - F(2) = 1 - 0.9 = 0.1$

(3) When a die is thrown, X denotes the no/. that turns up. Find $E(X)$, $E(X^2)$ & $Var(X)$.

Sol: X is a discrete random variable taking values 1, 2, 3, 4, 5, 6 & with probability $\frac{1}{6}$ for each.

x:	1	2	3	4	5	6
$P(x)$:	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \sum_i x_i P(x_i)$$
$$= \left(1\times\frac{1}{6}\right) + \left(2\times\frac{1}{6}\right) + \left(3\times\frac{1}{6}\right) + \left(4\times\frac{1}{6}\right) + \left(5\times\frac{1}{6}\right) + \left(6\times\frac{1}{6}\right)$$
$$= \frac{21}{6} = \frac{7}{2}$$
$$E(X^2) = \sum_i x_i^2 P(x_i) = \frac{1}{6}\left[1+4+9+16+25+36\right] = \frac{91}{6}$$
$$Var(X) = E[X^2] - [E(X)]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$$

(4) A random variable X has the following probability fun/.:

X:	0	1	2	3	4	5	6	7
$P(X)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(a) Find k. (b) Evaluate $P[X<6]$, $P[X\ge 6]$

(c) If $P[X \le C] > \frac{1}{2}$ find the minimum value of C.

Sol: (a) Since $\sum_i P(x_i) = 1$

(ie) $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow (10k-1)(10k+10) = 0$

$\Rightarrow k = \frac{1}{10}, -1$

Scanned with CamScanner

Since $P(X) \geq 0$ then we have $k = 1/10$.

(b) $P[X \geq 6] = P[X=6] + P[X=7]$

$$= 2k^2 + 7k^2 + k = 9k^2 + k = 9\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{9}{100} + \frac{1}{10} = \frac{9+10}{100} = \frac{19}{100}$$

$$P[X<6] = 1 - P[X \geq 6] = 1 - \frac{19}{100} = \frac{100-19}{100} = \frac{81}{100}$$

(c)

x	$P(x)$	$P[X \leq x]$
0	0	$P[X \leq 0] = P(x=0) = 0$
1	k	$P[X \leq 1] = 0 + k = k = 1/10$
2	$2k$	$P[X \leq 2] = 3k = 3/10$
3	$2k$	$P[X \leq 3] = 5k = 1/2$
4	$3k$	$P[X \leq 4] = 8k = 4/5$
5	k^2	$\cancel{P[X \leq 5] = 8k + k^2 = 8/10 + 1/100 = 81/100}$
$\cancel{6}$	$\cancel{2k^2}$	$\cancel{P[X \leq 6] =}$
$\cancel{7}$	$\cancel{7k^2+k}$	

$P[X \leq 4] = \frac{4}{5} > 1/2$. Hence 4 is the minimum value of c.

---

⑤ If X has the distribution fun. $F[x] = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \leq x < 4 \\ 1/2 & \text{for } 4 \leq x < 6 \\ 5/6 & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$

Find (i) The probability distribution of X.

(ii) $P(2 < X < 6)$ (v) $P(2 < X < 6 / X > 3)$

(iii) Mean of X

(iv) Variance of X.

<u>Sol:</u> (i) <u>Probability distribution of X:</u>

x	0	1	4	6	10
$P(x)$	0	$1/3$	$1/6$	$1/3$	$1/6$

① A r.v X has the following prob. dist.

x:	-2	-1	0	1	2	3
$P(x)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) k $(1/15)$ (ii) $P(x<2)$ $(1/2)$ (iii) $P(-2<x<2)$ $(2/5)$

② A discrete r.v X has the following prob. dist.

x:	0	1	2	3	4	5	6	7	8
$P(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find (i) a $(1/81)$ (ii) $P(0<x<3)$ $(8/81)$ (iii) $P(x \geq 3)$ $(8/9)$ (iv) the dist. fun. of x. (v) $P(1.5 < x < 4.5 / x > 2)$ $(2/9)$

(ii) $P(2 < X < 6) = P(X=4) = \frac{1}{6}$

(iii) Mean of $X = E[X] = \sum_i x_i P(x_i)$

28
4, 11, 17, 31, → CSE 22/12

$$= (0 \times 0) + \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{6}\right)$$

$$= \frac{1}{3} + \frac{2}{3} + 2 + \frac{5}{3} = \frac{8}{3} + 2 = \frac{14}{3}$$

(iv) $E[X^2] = \sum_i x_i^2 P(x_i) = (0 \times 0) + \left(1 \times \frac{1}{3}\right) + \left(16 \times \frac{1}{6}\right) + \left(36 \times \frac{1}{3}\right) + \left(100 \times \frac{1}{6}\right)$

$$= 0 + \frac{1}{3} + \frac{8}{3} + 12 + \frac{50}{3} = \frac{59}{3} + 12 = \frac{95}{3}$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{95}{3} - \left(\frac{14}{3}\right)^2 = \frac{95}{3} - \frac{196}{9} = \frac{285-196}{9}$$

$$= \frac{89}{9}$$

Scanned with CamScanner

(6) If $Var(x)=4$, find $Var(3x+8)$, where x is a random variable.

Sol: WKT $Var(ax+b)=a^2 Var(x)$

$$Var(3x+8)=9\,Var(x)=9\times 4=36$$

(7) X & Y are independent random variables with variance 2 & 3. Find the variance of $3x+4Y$.

Sol: Given $Var(x)=2$ & $Var(Y)=3$

$$Var(3x+4Y)=3^2 Var(x)+4^2 Var(Y)$$
$$=(9\times 2)+(16\times 3)=66$$

(8) If X be a random variable with $E(x)=1$ & $E[x(x-1)]=4$. Find $Var\,X$, $Var(2-3X)$ & $Var\left[\frac{X}{2}\right]$.

Sol: Given $E(x)=1$ & $E[x(x-1)]=4$

$$E[x(x-1)]=E[x^2-x]=E[x^2]-E[x]=E(x^2)-1$$
$$\Rightarrow 4=E(x^2)-1 \Rightarrow E[x^2]=5$$
$$Var(x)=E(x^2)-[E(x)]^2=5-1=4$$
$$Var(2-3x)=Var(2+(-3)x)=(-3)^2 Var(x)=9\times 4=36$$
$$Var\left(\frac{x}{2}\right)=\left(\frac{1}{2}\right)^2 Var(x)=\frac{1}{4}\times 4=1$$

(9) The probability func. of an infinite discrete distribution is given by $P[x=j]=\frac{1}{2^j}$, $j=1,2,\dots,\infty$. Find the mean & variance of the distribution. Also find $P[x \text{ is even}]$, $P[x\geq 5]$ & $P[x \text{ is divisible by } 3]$.

Sol: Given $P[x=j]=\frac{1}{2^j}$

j:	1	2	3	4	...
P(x=j):	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	...

$$E[x]=\sum_{j=1}^{\infty} x_j P(x_j)=(1)\left(\frac{1}{2}\right)+(2)\left(\frac{1}{2}\right)^2+(3)\left(\frac{1}{2}\right)^3+\dots$$
$$=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\dots\right]$$
$$=\frac{1}{2}\left[1-\frac{1}{2}\right]^{-2} \quad \left(\because (1-x)^{-2}=1+2x+3x^2+4x^3+\dots\right)$$
$$=\frac{4}{2}=2$$

$$E[x^2]=\sum_{j=1}^{\infty} x_j^2 P(x_j)=\sum_{j=1}^{\infty}(x_j^2+x_j-x_j)P(x_j)=\sum_{j=1}^{\infty}(x_j^2+x_j)P(x_j)-\sum_{j=1}^{\infty}x_jP(x_j)$$
$$=\sum_{j=1}^{\infty}x_j(x_j+1)P(x_j)-2=\left[(1)(2)\left(\frac{1}{2}\right)+(2)(3)\left(\frac{1}{2}\right)^2+(3)(4)\left(\frac{1}{2}\right)^3+\dots\right]-2$$
$$=\frac{1}{2}\left[1.2+2.3.\left(\frac{1}{2}\right)+3.4.\left(\frac{1}{2}\right)^2+\dots\right]-2$$
$$=\frac{2}{2}\left[1+3\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)^2+\dots\right]-2$$

$$= \left[1-\frac{1}{2}\right]^{-3} - 2 \qquad \left(\because (1-x)^{-3} = 1+3x+6x^2+10x^3+\cdots\right)$$

$$= 8-2 = 6$$

$$Var(X) = E[X^2] - [E(X)]^2 = 6-4 = 2$$

$$P[X \text{ is even}] = P[X=2] + P[X=4] + \cdots$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots \qquad \left[(1-x)^{-1} = 1+x+x^2+\cdots\right]$$

$$= \left[1-\frac{1}{4}\right]^{-1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

$$P[X \geq 5] = P[X=5] + P[X=6] + \cdots$$

$$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots = \left(\frac{1}{2}\right)^5 \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{1}{2}\right)^5 \left[1-\frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \times 2 = \frac{1}{16}$$

If the prob. mass func. of a rv X is given by $P(X=r) = kr^3$, $r = 1, 2, 3, 4$ find (i) k (ii) $P(\frac{1}{2} < X < \frac{7}{2} | X > 1)$ (iii) mean & variance of X (iv) distribution func. of X.

$$P[X \text{ is divisible by } 3] = P[X=3] + P[X=6] + \cdots$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots$$

$$= \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \cdots$$

$$= \left[1-\frac{1}{8}\right]^{-1} - 1 = \frac{8}{7} - 1 = \frac{1}{7}$$

Continuous Random Variables:

If X is a continuous random variable for any x_1 & x_2

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2).$$

Probability density func.:

For a continuous random variable X, a probability density func. is a func. such that (i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b for any a & b.

Cumulative distribution func.:

The cumulative distribution func. of a continuous random variable X is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$ for $-\infty < x < \infty$.

Result (i) $f(x) = \frac{d}{dx} F[x]$ $\qquad P(a<x<b) = F(b) - F(a)$

(ii) Mean $= \mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

(iii) $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

(iv) $Var(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= E(x^2) - [E(x)]^2$

(v) Standard deviation of $X = \sigma = \sqrt{Var(x)}$.

① If $f(x) = \begin{cases} kxe^{-x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$ is the p.d.f. of a r.v X. Find k. $(k=1)$

Problems:

① Given that the p.d.f. of a R.V. X is $f(x) = kx,\ 0<x<1$ find k & $P(x>0.5)$.

② If $f(x) = \begin{cases} c(4x-2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$ is the p.d.f. of a r.v X. Find c & $P(x>1)$ $(c = 3/8,\ P(x>1)=$

Sol: WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 kx\,dx = 1$

$\Rightarrow k\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow \frac{k}{2}(1-0) = 1 \Rightarrow k = 2$

$P(x>0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{1} kx\,dx = 2\left(\frac{x^2}{2}\right)_{0.5}^{1} = (1-0.25) = 0.75$

---

② A continuous random variable x has p.d.f. given by $f(x) = 3x^2,\ 0 \le x \le 1$. Find k such that $P(x>k) = 0.5$.

Sol: Given $f(x) = 3x^2,\ 0 \le x \le 1$

$P(x>k) = 0.5 \Rightarrow \int_k^{\infty} f(x)\,dx = 0.5 \Rightarrow \int_k^1 3x^2\,dx = 0.5$

$\Rightarrow 3\left(\frac{x^3}{3}\right)_k^1 = 0.5 \Rightarrow 1-k^3 = 0.5 \Rightarrow k^3 = 0.5$

$\Rightarrow k = (0.5)^{1/3} = 0.7937$

---

③ The p.d.f. of a continuous R.V. X is $f(x) = ke^{-|x|}$. Find k & the $F[x]$.

Sol: WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} ke^{-|x|}\,dx = 1$

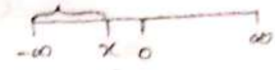
$\Rightarrow 2k\int_0^{\infty} e^{-x}\,dx = 1 \Rightarrow 2k\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$

$\Rightarrow -2k(0-1) = 1 \Rightarrow 2k = 1 \Rightarrow k = 1/2$

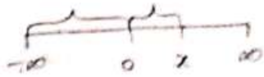
$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

Given $f(x) = ke^{-|x|} = \begin{cases} ke^{x}, & -\infty < x \le 0 \\ ke^{-x}, & 0 < x < \infty \end{cases} = \begin{cases} \frac{1}{2}e^{x}, & -\infty < x \le 0 \\ \frac{1}{2}e^{-x}, & 0 < x < \infty \end{cases}$

For $x \le 0$, $F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx = \frac{1}{2}\left(e^{x}\right)_{-\infty}^{x} = \frac{1}{2}\left(e^{x} - 0\right) = \frac{e^{x}}{2}$



For $x > 0$, $F(x) = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{x} \frac{1}{2}e^{-x}\,dx$



$$= \frac{1}{2}\left(e^{x}\right)_{-\infty}^{0} + \frac{1}{-2}\left(e^{-x}\right)_{0}^{x}$$

$$= \frac{1}{2}(1-0) + \frac{1}{-2}\left(e^{-x} - 1\right) = \frac{-1}{2}e^{-x} + 1 = 1 - \frac{1}{2}e^{-x}$$

$$\therefore F(x) = \begin{cases} \frac{e^{x}}{2}, & x \le 0 \\ 1 - \frac{1}{2}e^{-x}, & x > 0 \end{cases}$$

④ If X is a continuous R.V. with p.d.f. $f(x) = \begin{cases} x, & 0 \le x < 1 \\ \frac{3}{2}(x-1)^2, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$ find the cumulative distribution fun/. $F[x]$ of X & use it to find $P\left[\frac{3}{2} < X < \frac{5}{2}\right]$.

Sol: (i) If $x < 0$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,.dx = 0$

(ii) If $0 \le x < 1$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} x\,dx = \left(\frac{x^2}{2}\right)_{0}^{x} = \frac{x^2}{2}$

(iii) If $1 \le x < 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx \ne \cancel{\int_{1}^{x} f(x)\,dx} = \cancel{\frac{3}{2}\int_{1}^{x}(x-1)^2\,dx}$

$$= \cancel{\frac{3}{2}\left(\frac{(x-1)^3}{3}\right)_{1}^{x}} \ne \cancel{\frac{1}{2}(x-1)^3}$$

$$F(x) = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{x} \frac{3}{2}(x-1)^2\,dx$$

$$= \left(\frac{x^2}{2}\right)_{0}^{1} + \frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_{1}^{x} = \frac{1}{2} + \frac{1}{2}(x-1)^3$$

(iv) If $x \ge 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{2} \frac{3}{2}(x-1)^2\,dx + \int_{2}^{x} 0\,.dx$

$$= \left(\frac{x^2}{2}\right)_{0}^{1} + \frac{3}{2}\left(\frac{(x-1)^3}{3}\right)_{1}^{2} = \frac{1}{2} + \frac{1}{2}[1] = 1$$

$$\therefore F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & 0 \le x < 1 \\ \frac{1}{2} + \frac{1}{2}(x-1)^3, & 1 \le x < 2 \; ; \quad 1, \; x \ge 2 \end{cases}$$

$$P\left[\frac{3}{2}<x<\frac{5}{2}\right]=F\left[\frac{5}{2}\right]-F\left[\frac{3}{2}\right]$$

$$=1-\left(\frac{1}{2}+\frac{1}{2}\left(\frac{3}{2}-1\right)^3\right)=1-\left(\frac{1}{2}+\frac{1}{16}\right)$$

$$=1-\frac{9}{16}=\frac{7}{16}$$

(5) A continuous random variable X has the distribution funl. $F[x]=\begin{cases}0, & x\le 1\\ K(x-1)^4, & 1<x\le 3\\ 0, & x>3\end{cases}$ find K, probability density funl. $f(x)$, $P[X<2]$.

Sol: WKT $f(x)=\frac{d}{dx}F[x]$

$$\therefore f(x)=\begin{cases}0, & x\le 1\\ 4K(x-1)^3, & 1<x\le 3\\ 0, & x>3\end{cases}$$

WKT $\int_{-\infty}^{\infty} f(x)\,dx=1 \Rightarrow \int_1^3 4K(x-1)^3\,dx=1 \Rightarrow 4K\left(\frac{(x-1)^4}{4}\right)_1^3=1$

$$\Rightarrow K(2)^4=1 \Rightarrow K=\frac{1}{16}$$

$P[X<2]=F(2)=K(2-1)^4=\frac{1}{16}$ $\quad$ $P[X<2]=\int_1^2 \frac{1}{4}(x-1)^3dx=\frac{1}{4}\left[\frac{(x-1)^4}{4}\right]_1^2=\frac{1}{16}$

(6) Is the funl. defined as follows a density funl.? $f(x)=\begin{cases}0 & \text{for } x<2\\ \frac{1}{18}(3+2x) & \text{for } 2\le x\le 4\\ 0 & \text{for } x>4\end{cases}$

Sol: Condition for p.d.f. is $\int_{-\infty}^{\infty} f(x)\,dx=1$.

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{2}0.dx+\int_2^4\frac{1}{18}(3+2x)\,dx+\int_4^{\infty}0.dx$$

$$=\frac{1}{18}\left[3x+2\frac{x^2}{2}\right]_2^4=\frac{1}{18}\left(3x+x^2\right)_2^4=\frac{1}{18}\left[12+16-6-4\right]=1$$

Hence the given funl. is density funl..

(7) If the density funl. of a continuous R.V. X is given by $f(x)=\begin{cases}ax, & 0\le x\le 1\\ a, & 1\le x\le 2\\ 3a-ax, & 2\le x\le 3\\ 0, & \text{otherwise}\end{cases}$

(i) Find the value of a. (ii) the cumulative distribution funl. of X.
(iii) If x_1, x_2 & x_3 are 3 independent observations of X.
What is the probability that exactly one of these 3 is greater than 1.5?

Sol: (i) WKT, $\int_{-\infty}^{\infty} f(x)\,dx=1 \Rightarrow \int_0^1 ax\,dx+\int_1^2 a\,dx+\int_2^3(3a-ax)\,dx=1$

$$\Rightarrow a\left(\frac{x^2}{2}\right)_0^1+a(x)_1^2+\left(3ax-a\frac{x^2}{2}\right)_2^3=1$$

$$\Rightarrow \frac{a}{2}(1)+a(1)+\left(9a-\frac{9a}{2}-6a+2a\right)=1$$

$$\Rightarrow \frac{a}{2}+a+3a-\frac{5a}{2}=1 \Rightarrow 2a=1 \Rightarrow a=\frac{1}{2}$$

(ii) If $x < 0$ then $F(x) = 0$

If $0 \le x \le 1$ then $F[x] = \int_0^x ax\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{4}x^2$

If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 ax\,dx + \int_1^x a\,dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x = \frac{1}{4} + \frac{1}{2}(x-1)$$

$$= \frac{x}{2} + \frac{1}{4} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$$

If $2 \le x \le 3$ then $F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)\,dx$

$$F(x) = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left[3ax - a\frac{x^2}{2}\right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 3 + 1 = \frac{3}{4} - 2 + \frac{3x}{2} - \frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

If $x > 3$ then $F(x) = \int_0^1 \frac{1}{2}x\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx + \int_3^x 0\,.dx$

$$= \frac{1}{4} + \frac{1}{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_2^3 = \frac{1}{4} + \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - 3 + 1$$

$$\cancel{= -\frac{26}{4} + \frac{10}{2} = -\frac{13}{2} + 5 = -\frac{3}{2}} = -2 + 5 - 2 = 1$$

$$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{4} & , 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & , 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} & , 2 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

(iii) $P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx = \int_{1.5}^{2} \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx$

$$= \frac{1}{2}(x)_{1.5}^{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_2^3$$

$$= \frac{1}{2}(2 - 1.5) + \frac{9}{2} - \frac{9}{4} - 3 + 1 = \frac{1}{4} + \frac{9}{2} - \frac{9}{4} - 2 = \frac{9}{2} - 4 = \frac{1}{2}$$

Choosing an X & observing its value can be considered as a trial & $X > 1.5$ can be considered as a success. $\therefore p = \frac{1}{2}$, $q = \frac{1}{2}$

As we choose 3 independent observation of X, $n = 3$.

By Bernoulli's thm., P(exactly one value > 1.5) = P(1 success)

$$= {}^{n}C_x\, p^x q^{n-x} = 3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

① Let X be a continuous rv with pdf $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. Find (i) $P[X \le 0.4]$ (ii) $P(X > 3/4)$ (iii) $P(X > 1/2)$ (iv) $P(1/2 < X < 3/4)$ (v) $P(X > 3/4 \mid X > 1/2)$ (vi) $P(X < 3/4 \mid X > 1/2)$

② The diameter of an electric cable X is a continuous rv with pdf $f(x) = kx(1-x)$, $0 \le x \le 1$. Find (i) k (ii) cdf of X (iii) the value of a s.t. $P(X < a) = 2P(X > a)$ (iv) $P[X \le 1/2 \mid 1/3 < X < 2/3]$

⑧ A coin is tossed an infinite no. of times. If the probability of a head in a single toss is p, find the probability that kth head is obtained at the nth tossing but not earlier, with $q = 1-p$.

Sol: Given (i) A coin is tossed an infinite no. of times.

(ii) The probability of a head in a single toss is p.

(iii) $q = 1-p$.

k heads should be obtained at the nth tossing, but not earlier.

$\therefore$ (k-1) heads must be obtained in the first (n-1) tosses & 1 head at the nth toss.

Required probability = P[(k-1) heads in (n-1) tosses] × P[1 head in one toss]

$$= \left[(n-1)C_{(k-1)}\, p^{k-1} q^{n-k}\right] \times p$$

$$= (n-1)C_{(k-1)}\, p^{k} q^{n-k}$$

⑨ The sales of a convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution fun. of the following form:

$$F(x) = \begin{cases} 0, & x<0 \\ x^2/2, & 0\le x<1 \\ k(4x-x^2), & 1\le x<2 \\ 1, & x\ge 2 \end{cases}$$

Suppose that this convenience store's total sales on any given day are less than \$2000.

(i) Find the value of k.

(ii) Let A & B be the events that tomorrow the store's total sales are between 500 & 1500 dollars, & over 1000 dollars respectively. Find P(A) & P(B).

(iii) Are A & B independent events?

① A rv X has the pdf $f(x) = \begin{cases} 1/4, & -2<x<2 \\ 0, & \text{o.w.} \end{cases}$ Find $P(X<1)$, $P(|X|>1)$, $P(2X+3>5)$

② A cts rv X has pdf $f(x) = 3x^2, 0\le x\le 1$. Find a & b ∋: (i) $P(X\le a) = P(X>a)$ (ii) $P(X>b) = 0.05$

Sol: WKT $f(x) = \frac{d}{dx}F(x) = \begin{cases} 0, & x<0 \\ x, & 0\le x<1 \\ k(4-2x), & 1\le x<2 \\ 0, & x\ge 2 \end{cases}$

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 x\,dx + \int_1^2 k(4-2x)\,dx = 1$

$$\Rightarrow \left(\frac{x^2}{2}\right)_0^1 + k\left(4x-x^2\right)_1^2 = 1 \Rightarrow \frac{1}{2} + k(8-4-4+1) = 1$$

$$\Rightarrow \frac{1}{2} + k = 1 \Rightarrow k = \tfrac{1}{2}$$

(ii) $P(A) = P[500 < X < 1500] = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} x\,dx + \int_1^{1.5} \frac{1}{2}(4-2x)\,dx$

$$= \left(\frac{x^2}{2}\right)_{0.5}^{1} + \frac{1}{2}\left(4x-x^2\right)_1^{1.5} = 0.5 - 0.125 + \frac{1}{2}(3.75-3) = 0.75$$

$P(B) = P[X > 1000] = \int_1^2 \frac{1}{2}(4-2x)\,dx = \frac{1}{2}(4x-x^2)_1^2 = \frac{1}{2}(8-4-4+1)$

$$= \tfrac{1}{2}$$

(iii) $P(A\cap B) = P[1000 < X < 1500] = \int_{1}^{1.5} f(x)\,dx$

$$= \int_{1}^{1.5} \frac{1}{2}(4-2x)\,dx = \frac{1}{2}\left(4x - x^2\right)_{1}^{1.5} = \frac{1}{2}(3.75-4+1) = 0.375$$

$P(A).P(B) = (0.75)\left(\frac{1}{2}\right) = 0.375$

$\therefore P(A\cap B) = P(A).P(B)$

Hence A & B are independent.

* The probl. that a person will die in the time interval (t_1, t_2) is given by $P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\,dt$. The funl. $f(t)$ is determined from long t_1 records, & can be assumed to be $f(t) = \begin{cases} 3\times 10^{-9} t^2 (100-t)^2, & 0 \le t \le 100 \\ 0, & \text{o.e.w} \end{cases}$. Determine (i) the probl. that a person will die between the ages 60 & 70 & (ii) the probl. that he will die between those ages assuming he lived upto 60. $P(60 < x < 70 / x \ge 60)$

(10) Experience has shown that while walking in a certain park, the time X (in mins.), between seeing two people smoking has a density funl. of the form $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$ (i) Calculate the value of λ. (ii) Find the distribution funl. of X.

(iii) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? In atleast 7 minutes?

<u>Sol:</u> Given $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} \lambda x e^{-x}\,dx = 1 \Rightarrow \lambda\left[x\frac{e^{-x}}{-1} - e^{-x}\right]_{0}^{\infty} = 1$

$\Rightarrow \lambda(1) = 1 \Rightarrow \lambda = 1$

(ii) $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} xe^{-x}\,dx = \left[x\frac{e^{-x}}{-1} - e^{-x}\right]_{0}^{x}$

$= -xe^{-x} - e^{-x} + 1 = 1 - e^{-x}(x+1),\ x > 0$ $\quad F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{-x}(x+1), & x > 0 \end{cases}$

(iii) $P(2 < X < 5) = F(5) - F(2) = 1 - e^{-5}(6) - 1 + e^{-2}(3) = 0.3656$

$P(X \ge 7) = 1 - P(X < 7) = 1 - F(7) = 1 - 1 + e^{-7}(8) = 0.0073$

<u>Moments - Moment Generating Functions & their Properties:</u>

<u>Moments (Discrete case):</u>

Let X be a discrete R.V. taking the values $x_1, x_2, \ldots, x_n$ with probability mass funl. $p_1, p_2, \ldots, p_n$ respectively then the r^{th} moment about the origin is

μ_r' (about the origin) $= \sum_{i=1}^{n} x_i^r p_i$ —(1)

& μ_r' (about any point $x = A$) $= \sum_{i=1}^{n} (x_i - A)^r p_i$ —(2)

& μ_r (about mean) $= \sum_{i=1}^{n} (x_i - \text{mean})^r p_i$ —(3)

In particular from (1)

$\mu_1' = \sum_{i=1}^{n} x_i p_i = \text{Mean } (\bar{x})$

$\mu_2' = \sum_{i=1}^{n} x_i^2 p_i = \text{Mean square value}$

<u>Limitations of m.g.f:</u>

(1) A r.v X may have no moments although its m.g.f. exists.

(2) A r.v. X can have m.g.f. & some or all moments; yet the m.g.f. does not generate the moments.

(3) A r.v. X can have all or some moments, but m.g.f. does not exist except perhaps at one pt.

From ③, $\mu_2 = \sum_{i=1}^{n} (x_i - \text{mean})^2 p_i$ = Variance

$$= \mu_2' - (\mu_1')^2 \quad (\because \bar{x} = \mu_1')$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

A r.v. x has the following prob. distr.

x:	-2	-1	0	1	2	3
$P(x)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find (a) k (b) $P(x<2)$ & $P(-1<x<2)$ (c) cdf of x (d) mean (e) var (f) $P(1<x<3 / x>1)$

Moments (Continuous case):

If X is a continuous R.V. with p.d.f. $f(x)$ then defined in the interval (a,b).

$$\mu_r' = \int_a^b x^r f(x)\,dx$$

$$\mu_r' \text{ (about } A) = \int_a^b (x-A)^r f(x)\,dx$$

$$\mu_r \text{ (about mean)} = \int_a^b (x-\bar{x})^r f(x)\,dx$$

① A continuous r.v. X has the distribution funct. $F(x) = \begin{cases} 0, & x \le 1 \\ k(x-1)^4, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$ find k, pdf $f(x)$, $P(x<2)$

② Is the funct. defined as follows a density funct.? $f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$

Moment generating funct.: (m.g.f.).

The m.g.f. of a R.V. X (about origin) whose probability funct. $f(x)$ is given by

$$M_X(t) = E[e^{tX}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & \text{for a continuous probability distribution} \\ \sum_{x}^{\infty} e^{tx} P(x), & \text{for a discrete probability distribution} \end{cases}$$

where t is real parameter..

To find the r^{th} moment of X about origin, we know that

$$M_X(t) = E[e^{tX}] = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right]$$

$$= 1 + E\left[\frac{tX}{1!}\right] + E\left[\frac{t^2X^2}{2!}\right] + \cdots + E\left[\frac{t^rX^r}{r!}\right] + \cdots$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

(i) $M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$ —①

(ii) $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \quad (\text{using } \mu_r' = E(X^r))$

This gives the m.g.f. interms of moments. Thus the coeff. of $\frac{t^r}{r!}$ in $M_X(t)$ gives the r^{th} moment of the r.v. X about origin (μ_r'). Since $M_X(t)$ generates moments, it is known as moment generating funct..

Note: Diff. ① w.r.t. t, we get

$$M_X'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots \quad \text{—②}$$

Put $t=0$ in ②, we get $\boxed{\mu_1' = M_X'(0)}$

The first moment about origin is given by $\boxed{\mu_1' = M_X'(0) = \bar{x}}$, namely the mean.

Diff. ② w.r.t. t, we get

$$M_X''(t) = \mu_2' + t\mu_3' + \dots \quad \text{—③}$$

Put $t=0$ in ③, we get

$\boxed{M_X''(0) = \mu_2'}$. Hence the second moment about origin is given by $\boxed{\mu_2' = M_X''(0)}$

In general, we get $\mu_r' = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}$

<u>Note:</u> The m.g.f. of X about the pt $x=a$ is defined by

$$M_X(t) = E\left[e^{t(X-a)}\right] = E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + \dots + \frac{t^r}{r!}(X-a)^r + \dots\right]$$

$$= 1 + E[t(X-a)] + E\left[\frac{t^2}{2!}(X-a)^2\right] + \dots + E\left[\frac{t^r}{r!}(X-a)^r\right] + \dots$$

$$= 1 + tE(X-a) + \frac{t^2}{2!}E(X-a)^2 + \dots + \frac{t^r}{r!}E(X-a)^r + \dots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

Thus $[M_X(t)]_{x=a} = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$ where $\mu_r' = E[(X-a)^r]$

which gives the rth moment about the pt $x=a$.

<u>Properties of m.g.f.:</u>

① Let $Y = aX + b$, where X is a R.V. with m.g.f. $M_X(t)$. Then

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t(aX+b)}\right] = E\left[e^{tax} \cdot e^{bt}\right]$$

$$= e^{bt}E\left[e^{Xat}\right] = e^{bt}M_X(at)$$

② $M_{cX}(t) = E\left[e^{cXt}\right] = E\left[e^{X(ct)}\right] = M_X(ct)$ where c is a constant.

③ If X & Y are two independent random variables, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

<u>Proof:</u> $M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX+tY}\right] = E\left[e^{tX} \cdot e^{tY}\right]$

$$= E\left[e^{tX}\right] \cdot E\left[e^{tY}\right] \quad (\because X \,\&\, Y \text{ are independent})$$

$$= M_X(t) \cdot M_Y(t)$$

④ If $X_1, X_2, \dots, X_n$ are n independent RVs then

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \dots M_{X_n}(t)$$

<u>Sol:</u> $M_{X_1+X_2+\dots+X_n}(t) = E\left[e^{(X_1+X_2+\dots+X_n)t}\right] = E\left[e^{X_1t}\, e^{X_2t} \dots e^{X_nt}\right]$

$$= E\left[e^{X_1t}\right] E\left[e^{X_2t}\right] \dots E\left[e^{X_nt}\right] \quad (\because X_i\text{'s are independent})$$

$$= M_{X_1}(t)\, M_{X_2}(t) \dots M_{X_n}(t)$$

Problems:

① For the triangular distribution $f(x)=\begin{cases} x, & 0<x\le 1 \\ 2-x, & 1\le x<2 \\ 0, & \text{otherwise}\end{cases}$ find the mean, variance & the m.g.f.

Sol: Given $f(x)=\begin{cases} x, & 0<x\le 1 \\ 2-x, & 1\le x<2 \\ 0, & \text{otherwise}\end{cases}$

$$\text{Mean}=E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x^2\,dx+\int_1^2 x(2-x)\,dx$$

$$=\left(\frac{x^3}{3}\right)_0^1+\left(x^2-\frac{x^3}{3}\right)_1^2 = \frac{1}{3}+4-\frac{8}{3}-1+\frac{1}{3}=1$$

$$E[X^2]=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_0^1 x^3\,dx+\int_1^2 x^2(2-x)\,dx=\int_0^1 x^3\,dx+\int_1^2(2x^2-x^3)\,dx$$

$$=\left(\frac{x^4}{4}\right)_0^1+\left(\frac{2x^3}{3}-\frac{x^4}{4}\right)_1^2=\frac{1}{4}+\frac{16}{3}-4-\frac{2}{3}+\frac{1}{4}=\frac{1}{2}+\frac{14}{3}-4$$

$$=\frac{3+28-24}{6}=\frac{7}{6}$$

$$Var(X)=E(X^2)-[E(X)]^2=\frac{7}{6}-1=\frac{1}{6}$$

The m.g.f. of the r.v. X is

$$M_X(t)=E[e^{tX}]=\int_{-\infty}^{\infty} e^{tx}f(x)\,dx=\int_0^1 xe^{tx}\,dx+\int_1^2(2-x)e^{tx}\,dx$$

$$=\left[x\frac{e^{tx}}{t}-\frac{e^{tx}}{t^2}\right]_0^1+\left[(2-x)\frac{e^{tx}}{t}-(-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$=\frac{e^t}{t}-\frac{e^t}{t^2}+\frac{1}{t^2}+\frac{e^{2t}}{t^2}-\frac{e^t}{t}-\frac{e^t}{t^2}=\frac{e^{2t}}{t^2}-\frac{2e^t}{t^2}+\frac{1}{t^2}$$

$$=\frac{1}{t^2}[e^{2t}-2e^t+1]=\frac{1}{t^2}[e^t-1]^2$$

② Let X be a RV with probability law $P(X=r)=q^{r-1}p$; $r=1,2,3,\ldots$ Find the m.g.f. & hence mean & variance assume $p+q=1$.

Sol: WKT $M_X(t)=E[e^{tX}]=\sum_{x=1}^{\infty}e^{tx}P(x)=\sum_{r=1}^{\infty}e^{tr}P(r)$

$$=\sum_{r=1}^{\infty}e^{tr}q^{r-1}p=\frac{p}{q}\sum_{r=1}^{\infty}e^{tr}q^r=\frac{p}{q}\sum_{r=1}^{\infty}(qe^t)^r$$

$$=\frac{p}{q}\left[qe^t+(qe^t)^2+(qe^t)^3+\ldots\right]=\frac{p}{q}qe^t\left[1+qe^t+(qe^t)^2+\ldots\right]$$

$$=pe^t[1-qe^t]^{-1}$$

$$\therefore M_X(t)=\frac{pe^t}{1-qe^t}$$

$$M_X'(t)=\frac{(1-qe^t)pe^t-pe^t(-qe^t)}{(1-qe^t)^2}$$

$$\text{Mean}=M_X'(0)=\frac{(1-q)p-p(-q)}{(1-q)^2}=\frac{p-pq+pq}{(1-q)^2}=\frac{p}{p^2}=\frac{1}{p}\quad(\because p+q=1)$$

① Find the mgf of the rv X whose probability fun. $P(X=x)=\frac{1}{2^x}$, $x=1,2,\ldots$ Hence find its mean

② Find the probability distribution of the total no. of heads obtained in 3 tosses of a balanced coin. Hence obtain the mgf of X, mean of X & variance of X.

③ A rv X has density fun. given by $f(x)=\begin{cases}1/k & \text{for } 0<x<k \\ 0, & \text{otherwise}\end{cases}$ Find (i) mgf (ii) r-th moment (iii) mean (iv) variance

$$(1-qe^t)^4$$

$$M_x'(t) = \frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} = \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3} = \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$Var(x) = M_x''(0) - [M_x'(0)]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

③ The first 4 moments of a distribution about $X=4$ are 1, 4, 10 & 45 respectively. S.T. the mean is 5, variance is 3, $\mu_3 = 0$ & $\mu_4 = 26$.

Sol: Given that $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$ & $\mu_4' = 45$; $A = 4$

$$\mu_1 = \text{Mean} = A + \mu_1' = 4 + 1 = 5$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = 4 - \cancel{25}1 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 10 - 3(4)(1) + 2(1)^3 = 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$

$$= 45 - 40 + 24 - 3 = 26$$

④ Find the m.g.f. of an exponential r.v. & hence find its mean & variance.

Sol: The p.d.f. of an exponential distribution is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

④ Let x be a rv with pdf $f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
Find (i) $P(x>3)$ (ii) mgf of x (iii) $E(x)$ & $Var(x)$

⑤ Find the first 4 moments about the origin for a rv x having the pdf $f(x) = \begin{cases} \frac{4x(9-x^2)}{81}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$

The m.g.f. is given by

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x}\,dx = \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda}\right]_0^{\infty}$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{t-\lambda}\right]_0^{\infty} = \frac{\lambda}{t-\lambda}[0-1] = \frac{\lambda}{\lambda-t}$$

$\mu_1' = $ coeff. of $\frac{t}{1!}$

$\mu_2' = $ coeff. of $t^2/2!$

$$M_x(t) = \frac{\lambda}{\lambda - t} = \frac{\lambda}{\lambda(1-t/\lambda)} = [1-t/\lambda]^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots$$

$$\mu_1' = \frac{1}{\lambda} \; ; \; \mu_2' = \frac{2}{\lambda^2}$$

Mean $= M_x'(0)$

$$M_x'(t) = \frac{(\lambda-t)\cdot 0 - \lambda(-1)}{(\lambda-t)^2} = \frac{\lambda}{(\lambda-t)^2}$$

$$\therefore \text{Mean} = M_x'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$M_x''(t) = \frac{(\lambda-t)^2 \cdot 0 - \lambda 2(\lambda-t)\cdot(-1)}{(\lambda-t)^4} = \frac{2\lambda(\lambda-t)}{(\lambda-t)^4} = \frac{2\lambda}{(\lambda-t)^3}$$

The second moment $= M_X''(0) = \dfrac{2\lambda}{\lambda^3} = \dfrac{2}{\lambda^2}$

Variance $= M_X''(0) - [M_X'(0)]^2 = \dfrac{2}{\lambda^2} - \dfrac{1}{\lambda^2} = \dfrac{1}{\lambda^2}$

---

(5) The density fun. of a r.v. X is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, mean, variance & r^{th} moment.

Sol: Given $f(x) = kx(2-x)$, $0 \le x \le 2$ is a p.d.f.

WKT $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow k\int_0^2 x(2-x)\,dx = 1 \Rightarrow k\int_0^2 (2x - x^2)\,dx = 1$

$\Rightarrow k\left(x^2 - \dfrac{x^3}{3}\right)_0^2 = 1 \Rightarrow k\left(4 - \dfrac{8}{3}\right) = 1 \Rightarrow k = 3/4$

Mean $= E(X) = \displaystyle\int_{-\infty}^{\infty} x f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$

$= \dfrac{3}{4}\left[2\dfrac{x^3}{3} - \dfrac{x^4}{4}\right]_0^2 = \dfrac{3}{4}\left[\dfrac{16}{3} - 4\right] = 1$

$E[X^2] = \displaystyle\int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{3}{4}\int_0^2 x^3(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$

$= \dfrac{3}{4}\left[\dfrac{x^4}{2} - \dfrac{x^5}{5}\right]_0^2 = \dfrac{3}{4}\left[8 - \dfrac{32}{5}\right] = \dfrac{3}{4} \times \dfrac{8}{5} = \dfrac{6}{5}$

$Var(X) = E(X^2) - [E(X)]^2 = \dfrac{6}{5} - 1 = \dfrac{1}{5}$

$\mu_r' = E[X^r] = \displaystyle\int_{-\infty}^{\infty} x^r f(x)\,dx = \frac{3}{4}\int_0^2 x^r x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx$

$= \dfrac{3}{4}\left[2\dfrac{x^{r+2}}{r+2} - \dfrac{x^{r+3}}{r+3}\right]_0^2 = \dfrac{3}{4}\left[2\dfrac{(2)^{r+2}}{r+2} - \dfrac{(2)^{r+3}}{r+3}\right]$

$= \dfrac{3}{4}(2)^{r+3}\left[\dfrac{1}{r+2} - \dfrac{1}{r+3}\right] = 6(2)^r\left[\dfrac{r+3-r-2}{(r+2)(r+3)}\right]$

$= \dfrac{6\,2^r}{(r+2)(r+3)}$

---

(6) A continuous r.v. X has the p.d.f. $f(x)$ given by $f(x) = ce^{-|x|}$, $-\infty < x < \infty$. Find the value of c & m.g.f.

Sol: Given that $f(x) = ce^{-|x|}$ $-\infty < x < \infty$ is a p.d.f.

WKT $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} ce^{-|x|}\,dx = 1 \Rightarrow 2c\int_0^{\infty} e^{-x}\,dx = 1$

$\Rightarrow 2c\left(\dfrac{e^{-x}}{-1}\right)_0^{\infty} = 1 \Rightarrow -2c(0-1) = 1 \Rightarrow c = 1/2$

$\therefore f(x) = \dfrac{1}{2}e^{-|x|}$

Scanned with CamScanner

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx} e^{x} dx + \int_{0}^{\infty} e^{tx} e^{-x} dx\right] = \frac{1}{2}\left[\int_{-\infty}^{0} e^{(t+1)x} dx + \int_{0}^{\infty} e^{(t-1)x} dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{(t+1)x}}{t+1}\right)_{-\infty}^{0} + \left(\frac{e^{(t-1)x}}{t-1}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{(t+1)x}}{t+1}\right)_{\infty}^{0} + \left(\frac{e^{-(1-t)x}}{t-1}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t+1}(1-0) + \frac{1}{t-1}(0-1)\right] = \frac{1}{2}\left[\frac{1}{t+1} - \frac{1}{t-1}\right]$$

$$= \frac{1}{2}\left[\frac{t-1-t-1}{(t+1)(t-1)}\right] = \frac{-1}{t^2-1} = \frac{1}{1-t^2}$$

Binomial Distribution:

Bernoulli Trial:

Each trial has two possible outcomes, generally called success & failure. Such a trial is known as Bernoulli trial. The sample space for a Bernoulli trial is $S = \{s, f\}$.

E.g: (i) A toss of a single coin (head or tail)

(ii) The throw of a die (even or odd no.)

Binomial experiment:

An experiment consisting of a repeated no. of Bernoulli trials is called Binomial experiment. A binomial experiment must possess the following properties. (i) There must be a fixed no. of trials.

(ii) All trials must have identical probabilities of success (p).

(iii) The trials must be independent of each other.

Binomial distribution:

Consider a set of n independent Bernoullian trials (n being finite), in which the probability p of success in any trial is constant for each trial. Then $q = 1-p$ is the probability of failure in any trial. A r.v. X is said to follow binomial distribution if it assumes only non-(-)ve values & its probability mass fun. is given by

$$P(X=x) = p(x) = \begin{cases} {}^nC_x p^x q^{n-x}, & x = 0,1,2,\dots,n, \; q = 1-p \\ 0, & \text{otherwise} \end{cases}$$

The 2 independent constants n & p in the distribution are known as the parameters of the distribution. 'n' is also, sometimes known as the degree of the binomial distribution.

Binomial distribution: $P(X=x) = p(x) = nC_x p^x q^{n-x}$

The m.g.f. $M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx}\, nC_x p^x q^{n-x}$

$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$

$= nC_0 (pe^t)^0 q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + nC_n (pe^t)^n q^0$

$= q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + (pe^t)^n$

$= (pe^t + q)^n$

Mean $= E(X) = \left[\frac{d}{dt}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}(pe^t+q)^n\right]_{t=0}$

$= \left[n(pe^t+q)^{n-1} pe^t\right]_{t=0} = n(p+q)^{n-1} p = np \quad (\because p+q=1)$

$E(X^2) = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}\left(np(pe^t+q)^{n-1} e^t\right)\right]_{t=0}$

$= np\left[(pe^t+q)^{n-1} e^t + e^t (n-1)(pe^t+q)^{n-2} pe^t\right]_{t=0}$

$= np\left[(p+q)^{n-1} + (n-1)(p+q)^{n-2} p\right] = np\left[1+(n-1)p\right] = np + np^2(n-1)$

$= np + n^2p^2 - np^2$

$Var(X) = E(X^2) - [E(X)]^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p) = npq$

Problems:

① The mean & variance of a binomial variate are 8 & 6. Find $P(X \geq 2)$.

Sol: Given Mean $= np = 8$; Variance $= npq = 6$

$\frac{npq}{np} = \frac{6}{8} \Rightarrow q = 3/4$

$p = 1-q = 1-\frac{3}{4} = \frac{1}{4}$

$np = 8 \Rightarrow n/4 = 8 \Rightarrow n = 32$

$P(x) = nC_x p^x q^{n-x} = 32C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}$

$P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[32C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{32} + 32C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{31}\right]$

$= 1 - \left[\left(\frac{3}{4}\right)^{32} + 32\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{31}\right] = 1 - \left(\frac{3}{4}\right)^{31}\left(\frac{3}{4} + \frac{32}{4}\right)$

$= 1 - \frac{35}{4}\left(\frac{3}{4}\right)^{31}$

① The mean of a Binomial dist/. is 20 & S.D. is 4. Determine the parameters of the dist/.

② A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the prob. that there are (i) exactly 3 defectives (ii) not more than 3 defectives.

③ Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys & 2 girls (ii) atleast 1 boy (iii) atmost 2 girls & (iv) children of both genders. Assume equal probabilities for boys & girls.

② Find the probability that in tossing a fair coin 5 times, there will appear (a) 3 heads (b) 3 tails & 2 heads (c) atleast 1 head & (d) not more than 1 tail.

Let x denotes no/. of heads in 5 trials.

Sol: $p = \frac{1}{2}$, $q = \frac{1}{2}$ & $n = 5$

WKT $P(X=x) = nC_x p^x q^{n-x}$

(a) $P(\text{getting 3 heads}) = P(X=3) = 5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5\times4\times3}{1\times2\times3}\cdot\frac{1}{2^5} = \frac{5}{16}$

(b) We note that getting 3 tails & 2 heads is equivalent to getting 3 tails or 2 heads.

$P(\text{getting 3 tails \& 2 heads}) = P(\text{getting 2 heads})$

$= P(X=2) = 5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 10\cdot\frac{1}{2^5} = \frac{5}{16}$

(c) $P(\text{getting atleast 1 head}) = P(X\geq 1) = 1-P(X<1) = 1-P(X=0)$

$= 1-5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 1-\frac{1}{32} = \frac{31}{32}$.

(d) $P(\text{not more than 1 tail}) = P(\text{getting 0 tail}) + P(\text{getting 1 tail})$

$= P(\text{getting all heads}) + P(\text{getting 4 heads})$

$= 5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 + 5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1$

$= \frac{1}{2^5} + 5\cdot\frac{1}{2^5} = \frac{6}{2^5} = \frac{3}{16}$

---

③ An irregular 6 faced die is thrown such that the probability that it gives 3 even nos/. in 5 throws is twice the probability that it gives 2 even nos/. in 5 throws. How many sets of exactly 5 trials can be expected to give no even no/. out of 2500 sets.

Sol: WKT $P(X=x \text{ successes}) = nC_x p^x q^{n-x}$

Here p – the probability of getting an even no/. in a throw of a die.

Given $P(\text{getting 3 even nos/. in 5 throws}) = 2P(\text{getting 2 even nos/. in 5 throws})$

(ie) $P(X=3) = 2P(X=2)$

$5C_3 p^3q^2 = 2\times 5C_2 p^2 q^3$ (Here $n=5$)

$\Rightarrow 10p^3q^2 = 20p^2q^3 \Rightarrow p = 2q \Rightarrow p = 2(1-p) = 2-2p$

$\Rightarrow 3p = 2 \Rightarrow p = \frac{2}{3}$

$q = 1-p = 1-\frac{2}{3} = \frac{1}{3}$

$\therefore P(\text{getting no even no/.}) = P(X=0) = 5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5 = \frac{1}{3^5}$

In 2500 sets the no/. of sets having no even no/. is $= 2500\times P(X=0)$

$= 2500\times \frac{1}{3^5} = 10.2881$

⑤ In a certain town, 20% samples of the population is literate & assume that 200 investigators take samples of ten individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less literates in the samples?

Sol: Given P(an individual is literate) $= \frac{20}{100} = 0.2$

(ie) $p = 0.2$

$q = 1 - p = 1 - 0.2 = 0.8$ & $n = 10$ (sample size)

WKT $P(X=x) = nC_x \, p^x q^{n-x}$

P(an investigator reporting 3 or less as literate) $= P(X \le 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 10C_0(0.2)^0(0.8)^{10} + 10C_1(0.2)^1(0.8)^9 + 10C_2(0.2)^2(0.8)^8 + 10C_3(0.2)^3(0.8)^7$

$= (0.8)^7\left[0.512 + (10\times0.2\times0.64) + (45\times0.04\times0.8) + (120\times0.008\times1)\right]$

$= (0.8)^7(4.192) = 0.8791$

$\therefore$ (200 investigator reporting 3 or less as literate) $= 200\times0.8791 = 175.82$

---

(6) It is know that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 & offers a money-back guarantee that atmost 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Sol: Given $p = 0.01$; $q = 1-p = 1-0.01 = 0.09$; $n = 10$

WKT $P(X=x) = nC_x\, p^x q^{n-x}$

$\therefore P$(atmost 1 screw is defective) $= P(X\leq 1) = P(X=0) + P(X=1)$

$= 10C_0(0.01)^0(0.09)^{10} + 10C_1(0.01)^1(0.09)^9$

$= (0.09)^9(0.09 + (10\times0.01)) = (0.09)^9(0.19)$

$\therefore P$(a package will have to replace) $= 1 - P(X\leq 1) = 1 - (0.09)^9(0.19) = 1$

$\therefore$ 1% of the packages will have to replace.

---

(7) Suppose that the r.v. X is equal to the no. of hits obtained by a certain base ball player in his next 3 bats. If $P(X=1) = 0.3$, $P(X=2) = 0.2$ & $P(X=0) = 3P(X=3)$. Find $E(X)$.

Sol: WKT $P(X=0)+P(X=1)+P(X=2)+P(X=3) = 1$ — (1)

Given $P(X=0) = 3P(X=3)$ — (2)

Subs. (2) in (1) we get $3P(X=3)+0.3+0.2+P(X=3) = 1$

$\Rightarrow 4P(X=3) = 0.5 \Rightarrow P(X=3) = 0.125$

Given $P(X=0) = 3P(X=3) = 3\times0.125 = 0.375$

WKT $E(X) = \sum_i x_i P(x_i) = (1\times P(X=1)) + 2P(X=2) + 3P(X=3)$

$= 1(0.3) + 2(0.2) + 3(0.125) = 1.075$

---

(8) 6 dice are thrown 729 times. How many times do you expect atleast three dice to show a five or a six?

Sol: X - no. of dice showing 5 or 6.

p = probability of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$\therefore q = 1-p = 1-\frac{1}{3} = \frac{2}{3}$

P(atleast 3 dice showing 5 or 6) $= P(X\geq 3) = P(X=3)+P(X=4)+P(X=5) + P(X=6)$

$= 1 - P[X<3]$

$= 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + 6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1 + 6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0$

$= (20\times 8)\frac{1}{3^6} + (15\times 4)\frac{1}{3^6} + (6\times 2)\frac{1}{3^6} + \frac{1}{3^6}$

$= \frac{1}{3^6}(160+60+12+1) = \frac{233}{3^6}$

For 729 times, the expected no/. of times atleast 3 dice showing five or six $= N\times\frac{233}{3^6} = 729\times\frac{233}{3^6} = 233$ times

---

(9) The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge, find the probability that the bridge is destroyed?

Sol: Given P(hitting the target) $= \frac{1}{5}$ (ie) $p = \frac{1}{5}$

$q = 1-p = 1-\frac{1}{5} = \frac{4}{5}$; $n=6$

WKT $P(X=x) = nC_x\, p^x q^{n-x}$

P(the bridge is destroyed) $= P(X=2) = 6C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 = 15\times 4^4\times\frac{1}{5^6}$

$= 0.2458$

Poisson Distribution:

Poisson distribution is a limiting case of binomial distribution under the following assumptions.

(i) The no/. of trails 'n' should be indefinitely large. (ie) $n\to\infty$

(ii) The probability of successes 'p' for each trail is indefinitely small.

(iii) $np=\lambda$, should be finite where λ is a constant.

Derivation: Poisson distribution is given by $P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

The m.g.f. $M_X(t) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$

$= \sum_{x=0}^{\infty}\frac{(\lambda e^t)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$

$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right]$

$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$

Mean $= E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0} = \left[\frac{d}{dt}\left(e^{\lambda(e^t-1)}\right)\right]_{t=0}$

$$= \left[\frac{d}{dt}\left[e^{\lambda e^t} \cdot e^{-\lambda}\right]\right]_{t=0} = e^{-\lambda}\left[e^{\lambda e^t} \cdot \lambda e^t\right]_{t=0}$$

$$= e^{-\lambda} \lambda e^{\lambda} = \lambda$$

$$E[x^2] = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = e^{-\lambda}\left[\frac{d}{dt}\left[e^{\lambda e^t} \cdot \lambda e^t\right]\right]_{t=0} = \lambda e^{-\lambda}\left[\frac{d}{dt}\left[e^{\lambda e^t} \cdot e^t\right]\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda e^t} \cdot e^t + e^t \cdot e^{\lambda e^t} \cdot \lambda e^t\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda e^{\lambda}\right] = \lambda[1+\lambda] = \lambda + \lambda^2$$

$$Var(x) = E[x^2] - [E(x)]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Problems:

① If x is a Poisson variate such that $P(x=1) = \frac{3}{10}$ & $P(x=2) = \frac{1}{5}$, find $P(x=0)$ & $P(x=3)$.

Sol: WKT $P(x=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$P(x=1) = e^{-\lambda}\lambda = \frac{3}{10} \quad ; \quad P(x=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{1}{5} \Rightarrow e^{-\lambda}\lambda^2 = 2/5$$

$$\therefore \frac{e^{-\lambda}\lambda^2}{e^{-\lambda}\lambda} = \frac{2/5}{3/10} \Rightarrow \lambda = \frac{2}{5} \times \frac{10}{3} = 4/3$$

$$P(x=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = e^{-4/3} = 0.2636$$

$$P(x=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-4/3}(4/3)^3}{6} = e^{-4/3}\frac{4^3}{3^3 \times 6} = e^{-4/3}\left(\frac{32}{81}\right) = 0.1041$$

② One-fifth percent of the blades produced by a blade manufacturing factory turn out to be defective. The blades are in packets of 10. Use poisson distribution to calculate the approximate no. of packets containing (i) no defective (ii) one defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

x – no. of defective items in packets of 10.

Sol: Given $p = \frac{1/5}{100} = \frac{1}{500} = 0.002$, $n = 10$, $N = 10000$

$$Mean = \lambda = np = 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$$

The Poisson distribution is $P(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-0.02}(0.02)^x}{x!}$

(i) $P(\text{no defective}) = P(0) = \dfrac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02} = 0.9802$

$\therefore$ The total no. of packets containing no defective blades in a consignment of 10000 packets $= N \times P(\text{no defective}) = 10000 \times 0.9802 = 9802$ packets

(ii) $P(\text{one defective}) = P(1) = \dfrac{e^{-0.02}(0.02)^1}{1!} = 0.0196$

$\therefore$ No. of packets containing one defective $= N \times P(\text{one defective})$
$= 10000 \times 0.0196 = 196$ packets

(iii) $P(\text{two defective}) = P(2) = \dfrac{e^{-0.02}(0.02)^2}{2!} = 0.0002$

$\therefore$ No. of packets containing 2 defectives $= N \times P(2 \text{ defectives})$
$= 10000 \times 0.0002 = 2$ packets

---

(3) Six coins are tossed 6400 times. Using the poisson distribution, what is the approximate probability of getting six heads 10 times.

Sol: Given $n = 6400$ $\quad x$ – no. of times getting six heads.

Probability of getting one head with one coin $= \frac{1}{2}$

$\therefore$ The probability of getting six heads with six coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ Mean $= \lambda = np = 6400 \times \frac{1}{64} = 100$

The poisson distribution is $P(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-100}(100)^x}{x!}$

(ie) $P(\text{getting } x \text{ heads}) = \dfrac{e^{-100}(100)^x}{x!}$

$\therefore$ Probability of getting 6 heads 10 times $= P(X=10) = \dfrac{e^{-100}(100)^{10}}{10!}$

---

(4) If the m.g.f. of the r.v. X is $e^{4(e^t-1)}$, find $P(X=\mu+\sigma)$ where μ & σ^2 are the mean & variance of the poisson distribution.

Sol: The m.g.f. of a poisson distribution fun. $M_X(t) = e^{\mu(e^t-1)}$

where $\mu = \text{Mean} = 4$

Standard deviation $\sigma = \sqrt{\text{Variance}} = \sqrt{\text{Mean}}$ [Mean = Variance for a poisson distribution]

$\sigma = \sqrt{4} = 2$

$\therefore P(X=\mu+\sigma) = P(X=4+2) = P(X=6) = \dfrac{e^{-4}(4)^6}{6!} = 0.1042$

---

⑤ If X is a Poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$, find the variance.

Sol: The probability distribution for the poisson r.v. X is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots, \infty \ \& \ \lambda > 0$$

Given that $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

Dividing by $e^{-\lambda}\lambda^2$, we get

$$\frac{1}{2!} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!} \Rightarrow \frac{1}{2} = \frac{3}{8}\lambda^2 + \frac{1}{8}\lambda^4 \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1) = 0 \Rightarrow \lambda^2 = -4 \text{ (or) } \lambda^2 = 1 \Rightarrow \lambda = 1$$

$(\because \lambda > 0)$

For a poisson distribution, $Var(X) = \lambda = 1$

⑥ If X & Y are independent poisson variate such that $P(X=1) = P(X=2)$ & $P(Y=2) = P(Y=3)$ find the variance of $X - 2Y$.

Sol: WKT $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Given $P(X=1) = P(X=2) \Rightarrow \frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \Rightarrow \lambda = 2$

Also given $P(Y=2) = P(Y=3) \Rightarrow \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-\mu}\mu^3}{3!} \Rightarrow \mu = 3$

$Var(X) = 2 = \lambda$, $Var(Y) = \mu = 3$

$\therefore Var(X-2Y) = Var(X) + (-2)^2 Var(Y) = 2 + 4(3) = 14$

⑦ If X & Y are independent Poisson variates with means λ_1 & λ_2 respectively, find the probability that (i) $X+Y=k$, (ii) $X=Y$.

Sol: (i) WKT for a Poisson variate 'X'

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots \infty$$

$$\therefore P(X+Y=k) = \frac{e^{-(\lambda_1+\lambda_2)} \times (\lambda_1+\lambda_2)^k}{k!}$$ (By additive property of Poisson distribution)

(ii) $P(X=Y) = \sum_{r=0}^{\infty} P(X=r \cap Y=r) = \sum_{r=0}^{\infty} P(X=r) \cdot P(Y=r)$ ($\because X$ & Y are independent)

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^r}{r!} \cdot \frac{e^{-\lambda_2}\lambda_2^r}{r!} = e^{-(\lambda_1+\lambda_2)} \sum_{r=0}^{\infty} \frac{(\lambda_1\lambda_2)^r}{(r!)^2}$$

(8) The manufacturer of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 & guarantees that not more than 4 pins will be defective. What is the probability that a box will fail to meet the guaranteed quality?

Sol: Given $n=100$, $p=2\% = \frac{2}{100} = 0.02$

Mean $\lambda = np = 100 \times 0.02 = 2$

The poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}$

Now P(a box will fail to meet the guaranteed quality) $= P(X>4)$

$$= 1-P(X\le 4) = 1-[P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)]$$

$$= 1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}+\frac{e^{-2}2^3}{3!}+\frac{e^{-2}2^4}{4!}\right]$$

$$= 1-e^{-2}\left[1+2+2+\frac{4}{3}+\frac{2}{3}\right] = 1-e^{-2}(7) = 0.0527$$

(9) If X & Y are independent poisson r.v., show that the conditional distribution of X given X+Y is a binomial distribution.

Sol: Let X & Y are independent poisson R.V.'s with parameters λ_1 & λ_2 respectively.

Now $P(X=r / X+Y=n) = \frac{P(X=r \text{ and } X+Y=n)}{P(X+Y=n)}$ ~~P(X+Y). P(~~

$$= \frac{P(X=r \text{ and } Y=n-r)}{P(X+Y=n)} = \frac{P(X=r).P(Y=n-r)}{P(X+Y=n)}$$

$$= \frac{\frac{e^{-\lambda_1}\lambda_1^r}{r!}\cdot\frac{e^{-\lambda_2}\lambda_2^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

[$\because$ X is a poisson variate with parameter λ_1, Y is a poisson variate with parameter λ_2, X+Y is a poisson variate with parameter $\lambda_1+\lambda_2$]

$$= \frac{n!}{r!(n-r)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$$= nC_r\, p^r q^{n-r} \text{ where } p=\frac{\lambda_1}{\lambda_1+\lambda_2} \text{ \& } q=\frac{\lambda_2}{\lambda_1+\lambda_2}$$

which is a p.d.f. of a binomial distribution.

(10) The sum of two independent Poisson variates is a Poisson variate.

Sol: Let X_1, X_2 be the two independent Poisson variate with parameter λ_1, λ_2 respectively.

Now. $M_{x_1+x_2}(t) = M_{x_1}(t)\, M_{x_2}(t) = e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)}$

$= e^{(\lambda_1+\lambda_2)(e^t-1)} = e^{\lambda(e^t-1)}$

$\therefore$ The sum of 2 independent poisson variates is a poisson variate.

(11) If x_1 & x_2 are independent poisson variates show that $x_1 - x_2$ is not a poisson variate.

Sol. Given x_1 & x_2 are the two independent Poisson variates with parameter λ_1, λ_2 respectively.

Now $M_{x_1-x_2}(t) = M_{x_1}(t) \cdot M_{-x_2}(t) = M_{x_1}(t) \cdot M_{x_2}(-t)$

$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^{-t}-1)}$ which cannot be expressed in the form of $e^{\lambda(e^t-1)}$

$\therefore x_1 - x_2$ is not a poisson variate.

Geometric distribution:

A r.v. X is said to follow Geometric distribution, if it assumes only non-(-)ve values & its probability mass funl. is given by

$P(X=x) = (1-p)^{x-1} p = q^{x-1} p \ , \ x = 1, 2, \ldots \ , \ 0 < p \leq 1.$

The m.g.f. $M_x(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \sum_{x=1}^{\infty} pe^t (qe^t)^{x-1}$

$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t [1 + qe^t + (qe^t)^2 + \cdots]$

$= pe^t [1-qe^t]^{-1} = \frac{pe^t}{1-qe^t} = \frac{p}{e^{-t}-q}$

Mean $E(x) = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{p}{e^{-t}-q}\right)\right]_{t=0}$

$= \left[\frac{(e^{-t}-q).0 - p(e^{-t}.-1)}{(e^{-t}-q)^2}\right]_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p}$

$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{pe^{-t}}{(e^{-t}-q)^2}\right)\right]_{t=0}$

$= \left[\frac{(e^{-t}-q)^2\, pe^{-t}.-1 - pe^{-t}.\,2(e^{-t}-q).e^{-t}.-1}{(e^{-t}-q)^4}\right]_{t=0}$

$= \frac{-p^3 + 2p(1-q)}{(1-q)^4} = \frac{-p^3+2p^2}{p^4} = \frac{-p+2}{p^2} = \frac{-1}{p} + \frac{2}{p^2}$

$Var(x) = E(x^2) - [E(x)]^2 = \frac{-1}{p} + \frac{2}{p^2} - \frac{1}{p^2} = \frac{-1}{p} + \frac{1}{p^2} = \frac{-p+1}{p^2} = \frac{q}{p^2}$

Problems:

① If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6^{th} attempt.

Sol: Given $p=0.5$

$q=1-0.5=0.5$

WKT $P(X=x)=q^{x-1}p$

$P(X=6)=q^5 p=(0.5)^5(0.5)=0.0156$

② If the probability is 0.05 that a certain kind measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

Sol: Here $p=0.05$, $q=1-p=1-0.05=0.95$, $x=6$

WKT $P(X=x)=q^{x-1}p=(0.95)^5(0.05)=0.0387$

③ Let one copy of a magazine out of 10 copies bears a special prize following geometric random distribution. Determine its mean & variance.

Sol: Given $p=\frac{1}{10}$, $q=1-p=1-\frac{1}{10}=\frac{9}{10}$

Mean of the geometric distribution is $=\frac{1}{p}=10$

Variance $=\frac{q}{p^2}=\frac{9/10}{(1/10)^2}=\frac{9}{10}\times 10^2=90$

④ Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8.

(i) What is the probability that the target would be hit on 6^{th} attempt?

(ii) What is the probability that it takes him less than 5 shots?

(iii) What is the probability that it takes him an even no. of shots?

Sol: Given $p=0.8$, $q=1-p=1-0.8=0.2$

The geometric distribution is $P(X=x)=q^{x-1}p$, $x=1,2,\ldots$

(i) P(the target would be hit on the 6^{th} attempt) $=P(x=6)$

$=(0.2)^5(0.8)=0.000256$

(ii) P(it takes him less than 5 shots) $=P(X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$=(0.2)^0(0.8)+(0.2)^1(0.8)+(0.2)^2(0.8)+(0.2)^3(0.8)$

$=(0.8)+(0.2\times 0.8)+(0.2)^2(0.8)+(0.2)^3(0.8)$

$=0.9984$

(iii) P(it takes him an even no. of shots) $= P(x=2)+P(x=4)+P(x=6)+\cdots$

$$= (0.2)^{2-1}(0.8)+(0.2)^{4-1}(0.8)+(0.2)^{6-1}(0.8)+\cdots$$

$$= (0.2)(0.8)+(0.2)^3(0.8)+(0.2)^5(0.8)+\cdots$$

$$= (0.2)(0.8)\left[1+(0.2)^2+(0.2)^4+\cdots\right] = 0.16\left[1+0.04+(0.04)^2+\cdots\right]$$

$$= 0.16\left[1-0.04\right]^{-1} = 0.16\left[0.96\right]^{-1} = \frac{0.16}{0.96} = 0.1667$$

(5) Establish the memoryless property of geometric distribution.

Sol: If X has a geometric distribution, then for any two positive integers m & n, $P[X>m+n / X>m] = P(X>n)$

Proof: $$P[X>m+n / X>m] = \frac{P[X>m+n \cap X>m]}{P[X>m]} = \frac{P[X>m+n]}{P[X>m]}$$

Taking $P[X=r] = q^{r-1}p$, $r=1,2,3,\ldots$

$$P[X>k] = \sum_{r=k+1}^{\infty} q^{r-1}p = q^k p + q^{k+1}p + q^{k+2}p+\cdots$$

$$= q^k p\left[1+q+q^2+\cdots\right] = q^k p\left[1-q\right]^{-1} = q^k \quad (\because 1-q=p)$$

$\therefore P[X>m+n] = q^{m+n}$ & $P[X>m] = q^m$

$$\therefore P[X>m+n / X>m] = \frac{q^{m+n}}{q^m} = q^n$$

$$P[X>n] = q^n$$

Hence $P[X>m+n / X>m] = P[X>n]$

Uniform Distribution (or) Rectangular Distribution :

The p.d.f. of a uniform variable X in $(-a, a)$ is given by

$$f(x) = \begin{cases} \frac{1}{2a} &, -a < x < a \\ 0 &, \text{otherwise} \end{cases}$$

Derivation: $f(x) = \dfrac{1}{b-a}$, $a < x < b$

The m.g.f. $M_X(t) = \int_a^b e^{tx} f(x)\, dx = \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$

$$= \frac{1}{(b-a)t}\left[e^{tb} - e^{ta}\right]$$

$$= \frac{\left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \cdots\right] - \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots\right]}{(b-a)t}$$

$$= \frac{\frac{(b-a)t}{1!} + \frac{(b^2-a^2)t^2}{2!} + \frac{(b^3-a^3)t^3}{3!} + \cdots}{(b-a)t}$$

$$= 1 + \frac{(b+a)t}{2!} + \frac{(b^2+ba+a^2)}{3!}t^2 + \cdots$$

$$\text{Mean} = E(x) = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(1 + \frac{(b+a)t}{2!} + \frac{(b^2+ba+a^2)t^2}{3!} + \dots\right)\right]_{t=0}$$

$$= \left[\frac{b+a}{2} + \frac{(b^2+ba+a^2)2t}{3!} + \dots\right]_{t=0} = \frac{b+a}{2}$$

$$E[x^2] = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{b+a}{2} + \frac{(b^2+ba+a^2)2t}{6} + \dots\right)\right]_{t=0}$$

$$= \left[\frac{b^2+ba+a^2}{3} + \dots\right]_{t=0} = \frac{1}{3}(b^2+ba+a^2)$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = \frac{1}{3}(b^2+ba+a^2) - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{3}(b^2+ba+a^2) - \frac{1}{4}(a^2+b^2+2ab) = \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{1}{12}(a^2+b^2-2ab) = \frac{1}{12}(a-b)^2 = \frac{1}{12}(b-a)^2$$

Problems:

① Electric trains on a certain line run every half an hour between mid-night & six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait atleast 20 minutes?

Sol: Let x be the r.v. which denotes the waiting time for the next train. Assume that a man arrives at the station at random, the r.v. X is distributed uniformly in $(0, 30)$ with p.d.f. $f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore P(\text{atleast } 20 \text{ minutes}) = P(x \geq 20) = \int_{20}^{30} f(x)dx$$

$$= \frac{1}{30}\int_{20}^{30} dx = \frac{1}{30}(x)_{20}^{30} = \frac{1}{30}(30-20) = \frac{10}{30} = \frac{1}{3}$$

② S.T. for the uniform distribution $f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{elsewhere} \end{cases}$ the m.g.f. about the origin is $\frac{\sinh at}{at}$. Also, moments of even order are given by $\mu_{2n} = \frac{a^{2n}}{2n+1}$.

Sol: WKT the m.g.f. of uniform distribution in the interval (a, b) is

$$M_x(t) = \int_a^b e^{tx} f(x)dx$$

Here $f(x) = \frac{1}{2a}$ in $-a < x < a$

$$\therefore M_x(t) = \int_{-a}^{a} e^{tx}\frac{1}{2a}\,dx = \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)_{-a}^{a} = \frac{1}{2at}\left(e^{ta}-e^{-ta}\right)$$

$$= \frac{1}{at}\sinh at \quad \left(\because \frac{e^{x}-e^{-x}}{2} = \sinh x\right)$$

$$M_x(t) = \frac{1}{at}\sinh at = \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \dots\right] = 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \dots$$

Since there are no terms with odd powers of t in $M_x(t)$ all moments of odd order about origin vanish. (i.e) $\mu'_{2n+1} = 0$

In particular $\mu'_1 = 0 \Rightarrow$ Mean $= 0$

Thus $\mu_{2n} = \mu'_{2n}$ $(\because$ mean $= 0)$

$\therefore \mu_{2n+1} = 0$, $n = 0, 1, 2, \dots$

(ii) All moments of odd order about mean vanish. The moments of even order are given by $\mu_{2n} =$ coefficient of $\frac{t^{2n}}{(2n)!}$ in $M_x(t) = \frac{a^{2n}}{(2n+1)}$

③ If X is a r.v. uniformly distributed in $(0,1)$, find the p.d.f. of $Y = \sin X$. Also find the mean & variance of Y.

Sol: Given $Y = \sin X$. X has a uniform p.d.f. over $(0,1)$. $g(y) = 1$

$$G(y) = P(Y \le y) = P(\sin X \le y) = P(X \le \sin^{-1} y) = \int_0^{\sin^{-1}y} dx$$

$$= [x]_0^{\sin^{-1}y} = \sin^{-1} y$$

$$g(y) = \frac{d}{dy}(G(y)) = \frac{d}{dy}(\sin^{-1}y) = \frac{1}{\sqrt{1-y^2}}$$ where $0 < y < \sin 1$, 0 otherwise

$$\text{Mean} = E(Y) = \int_0^1 \sin x\,dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = -0.5403 + 1 = 0.4597$$

$$E(Y^2) = \int_0^1 \sin^2 x\,dx = \int_0^1 \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^1$$

$$= \frac{1}{2}\left[1 - \frac{\sin 2}{2}\right] = \frac{1}{2} - \frac{\sin 2}{4} = 0.2727$$

$$\cancel{Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \frac{\sin 2}{4} + \dots [1 - \cos 1]^2}$$

$$\cancel{= \frac{1}{2} - \frac{\sin 2}{4} - 1 - \cos^2 1 + 2\cos 1 = \frac{-1}{2}}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = 0.2727 - (0.4597)^2 = 0.0614$$

④ X is uniformly distributed with mean 1 & variance $\frac{4}{3}$, find $P(X<0)$.

Sol: Given that mean $= 1 \Rightarrow \frac{b+a}{2} = 1 \Rightarrow a+b = 2$ —①

Variance $= \frac{4}{3} \Rightarrow \cancel{\frac{(b-a)^2}{12}}\ \frac{(a-b)^2}{12} = 4/3 \Rightarrow (a-b)^2 = 16$ $\qquad \frac{(b-a)^2}{12} = 4/3$

$\Rightarrow a-b = 4$ —② $\qquad$ ② $\Rightarrow (b-a)^2 = 16$

$\Rightarrow b-a = 4$

Solving ① & ②, $2a = 6 \Rightarrow a = 3$, $b = -1$ $\qquad 2b = 6 \Rightarrow b = 3$

$a = -1$

$\therefore f(x) = \frac{1}{b-a} = \frac{1}{-1-3} = \frac{-1}{4}$ $\qquad f(x) = \frac{1}{b-a} = \frac{1}{3+1} = \frac{1}{4}$ $\qquad (-1, 3)$

$P(X<0) = \int_{-1}^{0} \frac{-1}{4}\,dx = \frac{-1}{4}(x)_{-1}^{0}$ $\qquad P(X<0) = \int_{-1}^{0} \frac{1}{4}\,dx = \frac{1}{4}(x)_{-1}^{0} = \frac{1}{4}$

$(-\alpha, \alpha), \alpha > 0$ find $\alpha \ni$: (i) $P(X>1) = 1/3$ (ii) $P(|X|<1) = P(|X|>1)$

⑤ A r.v. X has a uniform distribution over $(-3, 3)$ compute (i) $P(X<2)$, $P(|X|<2)$, $P(|X-2|<2)$ (ii) Find K for which $P(X>K) = \frac{1}{3}$.

Sol: WKT the p.d.f. of a r.v. X which is distributed uniformly in $(-a, a)$ is $f(x) = \begin{cases} \frac{1}{2a}, & -a<x<a \\ 0, & \text{otherwise} \end{cases}$

Here $a = 3$

$\therefore$ P.d.f. is $f(x) = \begin{cases} \frac{1}{6}, & -3<x<3 \\ 0, & \text{otherwise} \end{cases}$

(i) $P(X<2) = \int_{-3}^{2} f(x)\,dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{1}{6}(x)_{-3}^{2} = \frac{1}{6}(2+3) = \frac{5}{6}$

$P(|X|<2) = P(-2<X<2) = \int_{-2}^{2} f(x)\,dx = \frac{1}{6}(x)_{-2}^{2} = \frac{1}{6}(2+2) = \frac{2}{3}$

$P(|X-2|<2) = P(-2<X-2<2) = P(0<X<4) = \int_{0}^{3} f(x)\,dx$

$= \frac{1}{6}(x)_{0}^{3} = \frac{1}{6}(3) = \frac{1}{2}$

(ii) Given $P(X>K) = \frac{1}{3} \Rightarrow \int_{K}^{3} f(x)\,dx = \frac{1}{3} \Rightarrow \frac{1}{6}(x)_{K}^{3} = \frac{1}{3}$

$\Rightarrow 3-K = 2 \Rightarrow K = 1$

⑥ Buses arrive at a specified bus stop at 15 minutes intervals starting at 7a.m. that is 7a.m., 7.15a.m., 7.30a.m., etc. If a passenger arrives at the bus stop at a random time which is uniformly distributed between 7 & 7.30a.m. find the probability that he waits (a) less than 5 minutes (b) atleast 12 minutes for a bus.

Sol: Let X denote the no. of minutes past 7 that the passenger arrives at the bus stop. In the interval $(0, 30)$ X is a uniform r.v. & it follows that a passenger will have to wait less than 5 minutes if he arrives between

7.10 & 7.15 or between 7.25 & 7.30.

The p.d.f. is $f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$

(a) $P(10 \le X \le 15) + P(25 \le X \le 30) = \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$

$$= \frac{1}{30}\left[15-10+30-25\right] = \frac{10}{30} = \frac{1}{3}$$

(b) Passenger waits atleast 12 minutes (ie) he arrives between 7–7.03 or 7.15–7.18.

$P(\text{waiting time atleast 12 minutes}) = P(0 \le X \le 3) + P(15 \le X \le 18)$

$$= \int_0^3 \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx = \frac{1}{30}\left[3-0+18-15\right] = \frac{6}{30} = \frac{1}{5}$$

Exponential Distribution:

A continuous r.v. X is said to follow exponential distribution if its p.d.f. is given by, $f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \ge 0,\ \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$

Derivation: $f(x) = \lambda e^{-\lambda x},\ x \ge 0,\ \lambda > 0$

The m.g.f. $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx = \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx$

$$= \lambda\left[\frac{e^{(t-\lambda)x}}{t-\lambda}\right]_0^{\infty} = \frac{\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda}\left[0-1\right] = \frac{\lambda}{\lambda-t}$$

Mean $E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right]_{t=0}$

$$= \lambda\left[-1(\lambda-t)^{-2}(-1)\right]_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right)\right]_{t=0} = \lambda\left[(-2)(\lambda-t)^{-3}(-1)\right]_{t=0}$

$$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

✓ **Memoryless property of exponential distribution:**

If X is exponentially distributed, then $P(X > s+t \mid X > s) = P(X > t)$, for any $s, t > 0$.

Proof: $P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^{\infty} = -\left[0 - e^{-\lambda k}\right] = e^{-\lambda k}$ —— ①

Also, $P(X>s+t/X>s) = \dfrac{P(X>s+t \text{ and } X>s)}{P(X>s)}$

$$= \frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \text{ (by ①)} = e^{-\lambda t} = P(X>t)$$

Hence $P(X>s+t/X>s) = P(X>t)$

<u>Note</u>: The converse of this result is also true. (ie) If $P(X>s+t/X>s)=P(X>t)$ then X follows an exponential distribution.

<u>Problems</u>:

① The length of time a person speaks over phone follows exponential distribution with ~~xxxxx~~ parameter $\frac{1}{6}$. What is the probability that the person will talk for (i) more than 8 minutes (ii) between 4 & 8 minutes?

<u>Sol:</u> Given $f(x) = \frac{1}{6}e^{-x/6}$

(i) $P[X>8] = \int_8^\infty f(x)dx = \frac{1}{6}\int_8^\infty e^{-x/6}dx = \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_8^\infty$

$$= -\left[0 - e^{-4/3}\right] = e^{-4/3} = 0.2636$$

(ii) $P(4 \le X \le 8) = \int_4^8 \frac{1}{6}e^{-x/6}dx = \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_4^8 = -\left[e^{-4/3} - e^{-2/3}\right]$

$$= -\left[0.2636 - 0.5134\right] = 0.2498$$

---

② If X has an exponential distribution with parameter α, find the p.d.f. of $Y = \log X$.

<u>Sol:</u> $f_X(x) = \alpha e^{-\alpha y}$

$$f_Y(y) = \frac{d}{dy}F_Y(y)$$

$$F_Y(y) = P(Y \le y) = P(\log X \le y) = P(X \le e^y) = e^{-\alpha e^y}$$

$$f_Y(y) = \frac{d}{dy}\left[e^{-\alpha e^y}\right] = e^{-\alpha e^y}\left[-\alpha e^y\right] = -\alpha e^y e^{-\alpha e^y}$$

$$= \alpha e^y e^{-\alpha e^y}, \; -\infty < y < \infty$$

---

③ The time in hours required to repair a machine is exponentially distributed with ~~perimeter~~ parameter $\lambda = \frac{1}{2}$. (i) What is the probability that the repair time exceeds 2h? (ii) What is the conditional probability that a repair takes atleast 10h given that its duration exceeds 9h?

<u>Sol:</u> Given $\lambda = \frac{1}{2}$. Let X represents the time to repair the machine.

Then the density fun. of X is given by $f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2}, x \geq 0$

(i) $P(X>2) = \int_2^\infty \frac{1}{2} e^{-x/2} dx = \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^\infty = -\left[0 - e^{-1}\right] = e^{-1} = 0.3679$

(ii) The conditional probability that a repair takes atleast 10h given that its duration exceeds 9h is given by, $s+t=10 \therefore t=1$, $s=9$

$$P(X>10 \mid X>9) = P(X>9+1 \mid X>9) = P(X>1)$$

$(\because P(X>s+t \mid X>s) = P(X>t))$ by memoryless property

$$= \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^\infty = -\left[0 - e^{-1/2}\right]$$

$$= e^{-1/2} = 0.6065$$

④ If a continuous r.v. X follows uniform distribution in the interval $(0,2)$ & a continuous r.v. Y follows exponential distribution with parameter α, find α such that $P(X<1) = P(Y<1)$.

Sol: Since X follows uniform distribution over $(0,2)$, we get

$$f(x) = \begin{cases} \frac{1}{2-0}, & 0<x<2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}, & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$

Y follows exponential distribution $\therefore f(y) = \alpha e^{-\alpha y}, y \geq 0$

Given $P(X<1) = P(Y<1) \Rightarrow \int_0^1 f(x)dx = \int_0^1 f(y)dy$

$$\Rightarrow \int_0^1 \frac{1}{2} dx = \int_0^1 \alpha e^{-\alpha y} dy \Rightarrow \frac{1}{2}(x)_0^1 = \alpha\left(\frac{e^{-\alpha y}}{-\alpha}\right)_0^1$$

$$\Rightarrow \frac{1}{2} = -(e^{-\alpha} - 1) \Rightarrow \frac{1}{2} = (1 - e^{-\alpha})$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow -\alpha = \log_e 1/2 = \log_e 1 - \log_e 2 = 0 - \log_e 2$$

$$\Rightarrow \alpha = \log_e 2 = 0.6931$$

⑤ If X is exponentially distributed with parameter λ, find the value of k such that $\frac{P(X>k)}{P(X\leq k)} = a$.

Sol: Given $\frac{P(X>k)}{P(X\leq k)} = a \Rightarrow \frac{P(X>k)}{1-P(X>k)} = a \Rightarrow P(X>k) = a(1-P(X>k))$

$$\Rightarrow P(X>k)(1+a) = a$$

$$\Rightarrow P(X>k) = \frac{a}{1+a} \quad \text{—} ①$$

Since X is exponentially distributed with parameter λ, we get

$$P(X > K) = \int_K^\infty f(x)\,dx \quad \text{--- (2)}$$

Substituting (2) in (1), we get

$$\int_K^\infty f(x)\,dx = \frac{a}{1+a} \Rightarrow \int_K^\infty \lambda e^{-\lambda x}\,dx = \frac{a}{1+a} \Rightarrow \lambda\left(\frac{e^{-\lambda x}}{-\lambda}\right)_K^\infty = \frac{a}{1+a}$$

$$\Rightarrow -\left(0 - e^{-K\lambda}\right) = \frac{a}{1+a} \Rightarrow e^{-K\lambda} = \frac{a}{1+a}$$

$$\Rightarrow e^{K\lambda} = \frac{1+a}{a} \Rightarrow K\lambda = \log_e\left(\frac{1+a}{a}\right) \Rightarrow K = \frac{1}{\lambda}\log_e\left(\frac{1+a}{a}\right)$$

Two Dimensional Random Variables

Two dimensional random variable:

Let S be the sample space. Let $X = X(s)$ & $Y = Y(s)$ be two f... e assigning a real no/. to each outcome $s \in S$. Then (X, Y) is a two-dimensional random variable.

Two-dimensional discrete random variables:

If the possible values of (X, Y) are finite or countably infinite, then (X, Y) is called a two-dimensional discrete random variable. When (X, Y) is a two-dimensional discrete random variable the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Two-dimensional continuous random variables:

If (X, Y) can assume all values in a specified region R in the XY plane (X, Y) is called a two-dimensional continuous random variable.

Joint probability distribution:

The probabilities of the two events $A = \{X \le x\}$ & $B = \{Y \le y\}$ have defined as funs/. of x & y, respectively, called probability distribution funs/..

$F_X(x) = P(X \le x)$; $F_Y(y) = P(Y \le y)$

Joint probability distribution of two random variables X & Y:

The probability of the joint event $\{X \le x, Y \le y\}$, which is a fun/. of the nos/. x & y, by a joint probability distribution fun/ & denote it by the symbol $F_{X,Y}(x, y)$. Hence $F_{X,Y}(x, y) = p(X \le x, Y \le y)$.

Properties of the joint distribution:

1. $F_{X,Y}(-\infty, -\infty) = 0$; $F_{X,Y}(-\infty, y) = 0$ & $F_{X,Y}(x, -\infty) = 0$
2. $F_{X,Y}(\infty, \infty) = 1$
3. $0 \le F_{X,Y}(x, y) \le 1$
4. $F_{X,Y}(x, y)$ is a non-decreasing fun/. of x & y.
5. $F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_2, y_1)$
$$= P\{x_1 < X \le x_2 ; y_1 < Y \le y_2\} \ge 0$$
6. $F_{X,Y}(x, \infty) = F_X(x)$ & $F_{X,Y}(\infty, y) = F_Y(y)$

For a given fun/. to be a valid joint distribution fun/. of two dimensional RVs X & Y, it must satisfy the properties 1, 2 & 5.

Joint probability fun. of the discrete random variables X & Y:

If (X,Y) is a two-dimensional discrete r.v. such that $f(x_i, y_j) = P(X = x_i, Y = y_j) = P_{ij}$ is called the joint probability fun. or joint probability mass fun. of (X,Y) provided the following conditions are satisfied. (i) $P_{ij} \geq 0, \forall i \,\&\, j$ (ii) $\sum_j \sum_i P_{ij} = 1$

The set of triples $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ is called the joint probability distribution of (X,Y).

512, 516 → 19/02

Marginal probability distribution:

The individual probability distribution of a random variable is referred to as its marginal probability distribution. In general, the marginal probability distribution of X can be determined from the joint probability distribution of X & other random variables.

Marginal probability mass fun. of X:

If the joint probability distribution of two random variables X & Y is given, then the marginal probability fun. of X is given by

$$f(x) = P_X(x_i) = P(X = x_i)$$
$$= P[X = x_i, Y = y_1] + P[X = x_i, Y = y_2] + \cdots + P[X = x_i, Y = y_j] + \cdots + P[X = x_i, Y = y_m]$$
$$= p_{i1} + p_{i2} + \cdots + p_{ij} + \cdots + p_{im} = \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = P_{i\cdot}$$

<u>Note:</u> The set $\{x_i, P_{i\cdot}\}$ is called the marginal distribution of X.

Marginal probability mass fun. of Y:

If the joint probability distribution of two random variables X & Y is given, then the marginal probability fun. of Y is given by

$$f(y) = P_Y(y_j) = P(Y = y_j) = P_{\cdot j}$$

Here $p_{\cdot j} = \sum_{j=1}^{n} p_{ij} = p_{1j} + p_{2j} + \cdots + p_{nj}$

$$f(y) = P_Y(y_j) = P(Y = y_j)$$
$$= P[X = x_1, Y = y_j] + P[X = x_2, Y = y_j] + \cdots + P[X = x_i, Y = y_j] + \cdots + P[X = x_n, Y = y_j]$$
$$= p_{1j} + p_{2j} + \cdots + p_{ij} + \cdots + p_{nj} = \sum_{i=1}^{n} p_{ij} = P_{\cdot j}$$

<u>Note:</u> The set $\{y_j, p_{\cdot j}\}$ is called the marginal distribution of Y.

Conditional probability distribution:

$$P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i \,\&\, Y = y_j\}}{P(Y = y_j)} = \frac{P_{ij}}{P_{\cdot j}}$$

is called the conditional probability fun. of X, given $Y = y_j$. The collection of pairs $\left\{x_i, \frac{P_{ij}}{P_{\cdot j}}\right\}$, $i = 1, 2, \ldots$ is called the conditional probability distribution of X, given $Y = y_j$. Similarly, the collection of pairs, $\left\{y_j, \frac{P_{ij}}{P_{i\cdot}}\right\}$, $j = 1, 2, \ldots$ is called the conditional probability distribution of Y given $X = x_i$.

Let (X,Y) be the two dimensional continuous r.v.. The conditional p.d.f. of X given Y is denoted by $f(x/y)$ & is defined as $f(x/y) = \frac{f(x,y)}{f(y)}$.

Similary, the conditional p.d.f. of Y given X is denoted by $f(y/x)$ & is defined as, $f(y/x) = \frac{f(x,y)}{f(x)}$.

Independent random variables:

Two RVs X & Y are said to be independent if $f(x,y) = f(x).f(y)$ where $f(x,y)$ is the joint p.d.f. of (X,Y), $f(x)$ is the marginal density fun/. of X & $f(y)$ is the marginal density fun/. of Y.

The r.vs X & Y are said to be independent r.vs if $P_{ij} = P_{i\cdot} \times P_{\cdot j}$ where P_{ij} is the joint probability fun/. of (X,Y), $P_{i\cdot}$ is the marginal probability fun/. of X & $P_{\cdot j}$ is the marginal probability fun/. of Y.

Joint probability density fun/.:

If (X,Y) is a two-dimensional continuous r.v. such that $P\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\} = f(x,y)\,dx\,dy$, then $f(x,y)$ is called the joint p.d.f. of (X,Y), provided $f(x,y)$ satisfies the following conditions.

(i) $f(x,y) \ge 0$, $\forall (x,y) \in R$, where R is the range space.

(ii) $\iint_R f(x,y)\,dx\,dy = 1$

In particular, $P(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f(x,y)\,dx\,dy$

Cumulative distribution fun/.:

If (X,Y) is a two-dimensional continuous r.v., then $F(x,y) = P(X \le x \,\&\, Y \le y)$ is called the c.d.f. of (X,Y) & is defined as, $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy$.

Marginal density fun/.:

If (X,Y) is a two-dimensional continuous r.v. such that $P\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, -\infty < Y < \infty\} = \int_{-\infty}^{\infty} \int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(x,y)\,dy\,dx$.

Let (X,Y) be the two dimensional r.v.. Then the marginal p.d.f. of X is denoted by $f(x)$ & is defined as, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

Similarly the marginal p.d.f. of Y is denoted by $f(y)$ & is defined as, $f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

Joint probability density funⁿ:

Let (X,Y) be the two dimensional r.v. & $F(x,y)$ be the joint probability distribution funⁿ. Then the joint p.d.f. of X & Y is denoted by $f(x,y)$ & is defined as, $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x\, \partial y}$.

Problems:

① The joint probability mass funⁿ of (X,Y) is given by $P(x,y) = K(2x+3y)$, $x = 0,1,2$; $y = 1,2,3$. Find all the marginal & conditional probability distributions. Also find the probability distribution of $(X+Y)$ & $P(X+Y>3)$.

Sol:

X \ Y	1	2	3
0	3K	6K	9K
1	5K	8K	11K
2	7K	10K	13K

$\sum_{j=1}^{3} \sum_{i=1}^{3} P(x_i, y_j) = 1$

$\Rightarrow 72K = 1 \Rightarrow K = 1/72$

X \ Y	1	2	3	$P(X=x_i)$ $P_X(x) = f(x) = P_{i\cdot}$
0	$3/72$	$6/72$	$9/72$	$P(X=0) = 18/72$ $P_{11}+P_{12}+P_{13}$
1	$5/72$	$8/72$	$11/72$	$P(X=1) = 24/72$
2	$7/72$	$10/72$	$13/72$	$P(X=2) = 30/72$
$P(Y=y_j)$ $P_Y(y) = f(y)$ $= P_{\cdot j}$	$P_{11}+P_{21}+P_{31}$ $P(Y=1) = \frac{15}{72}$	$P(Y=2) = \frac{24}{72}$	$P(Y=3) = \frac{33}{72}$	

56, 60, 61, 64, 11, → 21/02

Marginal distributions of X:

$P(X=0) = \frac{18}{72}$; $P(X=1) = \frac{24}{72}$; $P(X=2) = \frac{30}{72}$

Marginal distributions of Y:

$P(Y=1) = \frac{15}{72}$; $P(Y=2) = \frac{24}{72}$; $P(Y=3) = \frac{33}{72}$

Conditional distribution of X, given Y is $P\{X = x_i / Y = y_j\}$

$P(X=0/Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{3/72}{15/72} = \dfrac{1}{5}$

$P(X=1/Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{5/72}{15/72} = \dfrac{1}{3}$

$P(X=2/Y=1) = \dfrac{P(X=2, Y=1)}{P(Y=1)} = \dfrac{7/72}{15/72} = \dfrac{7}{15}$

$P(X=0/Y=2) = \dfrac{P(X=0, Y=2)}{P(Y=2)} = \dfrac{6/72}{24/72} = \dfrac{1}{4}$

$P(X=1/Y=2) = \dfrac{P(X=1, Y=2)}{P(Y=2)} = \dfrac{8/72}{24/72} = \dfrac{1}{3}$

$P(X=2/Y=2) = \dfrac{P(X=2, Y=2)}{P(Y=2)} = \dfrac{10/72}{24/72} = \dfrac{5}{12}$

$P(X=0/Y=3) = \dfrac{P(X=0, Y=3)}{P(Y=3)} = \dfrac{9/72}{33/72} = \dfrac{9}{33} = \dfrac{3}{11}$

$P(X=1/Y=3) = \dfrac{P(X=1, Y=3)}{P(Y=3)} = \dfrac{11/72}{33/72} = \dfrac{1}{3}$

$P(X=2/Y=3) = \dfrac{P(X=2, Y=3)}{P(Y=3)} = \dfrac{13/72}{33/72} = \dfrac{13}{33}$

Conditional distribution of Y, given X is $P\{Y=y_j \mid X=x_i\}$

$$P(Y=1/X=0)=\frac{P(X=0,Y=1)}{P(X=0)}=\frac{3/72}{18/72}=\frac{1}{6}$$

$$P(Y=3/X=1)=\frac{P(X=1,Y=3)}{P(X=1)}=\frac{11/72}{24/72}=\frac{11}{24}$$

$$P(Y=2/X=0)=\frac{P(X=0,Y=2)}{P(X=0)}=\frac{6/72}{18/72}=\frac{1}{3}$$

$$P(Y=1/X=2)=\frac{P(X=2,Y=1)}{P(X=2)}=\frac{7/72}{30/72}=\frac{7}{30}$$

$$P(Y=3/X=0)=\frac{P(X=0,Y=3)}{P(X=0)}=\frac{9/72}{18/72}=\frac{1}{2}$$

$$P(Y=2/X=2)=\frac{P(X=2,Y=2)}{P(X=2)}=\frac{10/72}{30/72}=\frac{1}{3}$$

$$P(Y=1/X=1)=\frac{P(X=1,Y=1)}{P(X=1)}=\frac{5/72}{24/72}=\frac{5}{24}$$

$$P(Y=3/X=2)=\frac{P(X=2,Y=3)}{P(X=2)}=\frac{13/72}{30/72}=\frac{13}{30}$$

$$P(Y=2/X=1)=\frac{P(X=1,Y=2)}{P(X=1)}=\frac{8/72}{24/72}=\frac{1}{3}$$

Probability distribution of X+Y:

$X+Y$	Probability
1	$P(0,1)=3/72$
2	$P(0,2)+P(1,1)=\frac{6}{72}+\frac{5}{72}=\frac{11}{72}$
3	$P(0,3)+P(1,2)+P(2,1)=\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$
4	$P(1,3)+P(2,2)=\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5	$P(2,3)=\frac{13}{72}$

$$P[X+Y>3]=P[X+Y=4]+P[X+Y=5]=\frac{21}{72}+\frac{13}{72}=\frac{34}{72}$$

(2) The joint probability mass funl. (p.m.f.) of X & Y is

X \ Y	0	1	2
0	0.1	0.04	0.02
1	0.08	0.2	0.06
2	0.06	0.14	0.3

Compute the marginal p.m.f. of X & Y, $P[X\le 1, Y\le 1]$ & check if X & Y are independent.

Sol:

X \ Y	0	1	2	$P(X=x_i)=P_{i\cdot}$
0	0.1	0.04	0.02	$P(X=0)=0.16$
1	0.08	0.2	0.06	$P(X=1)=0.34$
2	0.06	0.14	0.3	$P(X=2)=0.5$
$P(Y=y_j)=P_{\cdot j}$	$P(Y=0)=0.24$	$P(Y=1)=0.38$	$P(Y=2)=0.38$	

The marginal p.m.f. of X are $P(X=0)=0.16$; $P(X=1)=0.34$ & $P(X=2)=0.5$;

The marginal p.m.f. of Y are $P(Y=0)=0.24$; $P(Y=1)=0.38$ & $P(Y=2)=0.38$

Now, $P[X\le 1, Y\le 1] = P[X=0, Y=0] + P[X=0, Y=1] + P[X=1, Y=0] + P[X=1, Y=1]$

$$= 0.1 + 0.04 + 0.08 + 0.2 = 0.42$$

If $P_{ij} = P_{i\cdot} \times P_{\cdot j}$ then we can say that X & Y are independent.

We have $P_{0\cdot} = 0.16$ & $P_{\cdot 0} = 0.24$

$\therefore P_{0\cdot} \times P_{\cdot 0} = 0.0384 \neq 0.1 = P_{00}$

$\therefore P_{ij} \neq P_{i\cdot} \times P_{\cdot j}$

Hence X & Y are not independent.

(3) Suppose the joint p.d.f is given by $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) ; & 0\le x\le 1, 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$. Obtain the marginal p.d.f. of X & that of Y. Hence or otherwise find $P\left[\frac{1}{4}\le y\le \frac{3}{4}\right]$.

Sol: Given that $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0\le x\le 1, 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$

The marginal p.d.f. of X is $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{6}{5}\int_0^1 (x+y^2)\,dy$

$$= \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_0^1 = \frac{6}{5}\left[x+\frac{1}{3}\right], \quad 0\le x\le 1$$

The marginal p.d.f. of Y is $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{6}{5}\int_0^1 (x+y^2)\,dx$

$$= \frac{6}{5}\left[\frac{x^2}{2} + y^2 x\right]_0^1 = \frac{6}{5}\left[\frac{1}{2}+y^2\right], \quad 0\le y\le 1$$

$$P\left[\frac{1}{4}\le y\le \frac{3}{4}\right] = \int_{1/4}^{3/4} f(y)\,dy = \int_{1/4}^{3/4} \frac{6}{5}\left(\frac{1}{2}+y^2\right)dy = \frac{6}{5}\left[\frac{1}{2}y + \frac{y^3}{3}\right]_{1/4}^{3/4}$$

$$= \frac{6}{5}\left[\frac{3}{8} + \frac{9}{64} - \frac{1}{8} - \frac{1}{192}\right] = \frac{6}{5}\left[\frac{1}{4} + \frac{26}{192}\right] = \frac{6}{5}\left[\frac{48+26}{192}\right]$$

$$= \frac{6}{5}\times\frac{74}{192} = 0.4625$$

(4) Let X & Y have joint p.d.f. $f(x,y) = 2$, $0<x<y<1$. Find the m.d.f. find the conditional density funct. of Y given $X=x$.

Sol: The marginal density funct. of X is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2(y)_x^1 = 2(1-x), \quad 0<x<1$$

The marginal density funct. of Y is given by

$\therefore f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 2\,dx = 2(x)_0^y = 2y,\ 0<y<1.$

The conditional density funf. of Y given X=x is,

$f(Y/X) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$

⑤ If the joint p.d.f. of a 2 dimensional r.v. (X,Y) is given by $f(x,y) = \begin{cases} 2, & 0<y<x<1 \\ 0, & \text{otherwise} \end{cases}$
Find the marginal density funf. of X & Y. Also find X & Y are independent.

Sol: The marginal density funf. of X is given by

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 2\,dy = 2(y)_0^x = 2x,\ 0<x<1$

The marginal density funf. of Y is given by

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 2\,dx = 2(x)_y^1 = 2(1-y),\ 0<y<1$

$f(x).f(y) = (2x)(2(1-y)) = 4x(1-y) \neq 2 = f(x,y)$

$\therefore f(x,y) \neq f(x).f(y)$. Hence the rvs X & Y are dependent on each other.

⑥ The joint p.d.f. of the r.v. (X,Y) is given by $f(x,y) = kxye^{-(x^2+y^2)},\ x>0, y>0$.
Find the value of k & prove also that X & Y are independent.

Sol: WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}\,dx\,dy = 1$

$\Rightarrow k\int_0^{\infty}\int_0^{\infty} xye^{-x^2}e^{-y^2}\,dx\,dy = 1$

Put $x^2 = t$
$2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$
when $x=0,\ t=0$
$x\to\infty,\ t\to\infty$

$\Rightarrow k\int_0^{\infty}\int_0^{\infty} e^{-t}\frac{dt}{2}\,ye^{-y^2}\,dy = 1$

$\Rightarrow \frac{k}{2}\int_0^{\infty}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} ye^{-y^2}\,dy = 1 \Rightarrow -\frac{k}{2}\int_0^{\infty}(0-1)\,ye^{-y^2}\,dy = 1$

Put $y^2 = u$
$2y\,dy = du \Rightarrow y\,dy = \frac{du}{2}$
when $y=0,\ u=0$
$y\to\infty,\ u\to\infty$

$\Rightarrow \frac{k}{2}\int_0^{\infty} ye^{-y^2}\,dy = 1 \Rightarrow \frac{k}{2}\int_0^{\infty} e^{-u}\frac{du}{2} = 1$

$\Rightarrow \frac{k}{4}\left(\frac{e^{-u}}{-1}\right)_0^{\infty} = 1 \Rightarrow -\frac{k}{4}(0-1) = 1 \Rightarrow k/4 = 1 \Rightarrow k = 4$

15, 39, 46, 49, 54

The marginal density funf. of X is given by

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} 4xye^{-(x^2+y^2)}\,dy = 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}\,dy$

$= 4xe^{-x^2}\int_0^{\infty} e^{-u}\frac{du}{2} = 2xe^{-x^2}\left(\frac{e^{-u}}{-1}\right)_0^{\infty} = 2xe^{-x^2},\ x>0$

The marginal density fun/. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^{\infty} 4xye^{-(x^2+y^2)}\,dx=4ye^{-y^2}\int_0^{\infty} xe^{-x^2}\,dx$$

$$=4ye^{-y^2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}=2ye^{-y^2}\left(\frac{e^{-t}}{-1}\right)_0^{\infty}=2ye^{-y^2},\ y>0$$

Now, $f(x)\,f(y)=2xe^{-x^2}\cdot 2ye^{-y^2}=4xye^{-(x^2+y^2)}=f(x,y)$

$\therefore$ X & Y are independent.

(7) Given $f_{xy}(x,y)=\begin{cases} cx(x-y), & 0<x<2,\ -x<y<x \\ 0, & \text{elsewhere}\end{cases}$ (a) Evaluate c (b) Find $f_x(x)$

(c) $f_{Y/X}(y/x)$ & (d) $f_y(y)$.

Sol: (a) WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1 \Rightarrow \int_0^2\int_{-x}^{x} cx(x-y)\,dy\,dx=1$

$$\Rightarrow c\int_0^2 x\left(xy-\frac{y^2}{2}\right)_{-x}^{x}dx=1 \Rightarrow c\int_0^2 x\left(x^2-\frac{x^2}{2}+x^2+\frac{x^2}{2}\right)dx=1$$

$$\Rightarrow c\int_0^2 2x^3\,dx=1 \Rightarrow 2c\left(\frac{x^4}{4}\right)_0^2=1 \Rightarrow \frac{c}{2}(16)=1 \Rightarrow c=1/8$$

(b) Marginal density fun/. of X is given by

$$f_x(x)=f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\frac{1}{8}\int_{-x}^{x} x(x-y)\,dy=\frac{x}{8}\left(xy-\frac{y^2}{2}\right)_{-x}^{x}$$

$$=\frac{x}{8}\left(x^2-\frac{x^2}{2}+x^2+\frac{x^2}{2}\right)=\frac{x^3}{4},\ 0<x<2$$

(c) $f_{Y/X}(y/x)=\dfrac{f(x,y)}{f(x)}=\dfrac{\frac{1}{8}x(x-y)}{x^3/4}=\dfrac{1}{2x^2}(x-y),\ -x<y<x$

(d) Marginal density fun/. of Y is given by

$$f_Y(y)=f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\frac{1}{8}\int_0^2 x(x-y)\,dx=\frac{1}{8}\int_0^2 (x^2-xy)\,dx$$

$$=\frac{1}{8}\left(\frac{x^3}{3}-\frac{x^2y}{2}\right)_0^2=\frac{1}{8}\left(\frac{8}{3}-2y\right)=\frac{1}{24}(8-6y)=\frac{1}{12}(4-3y),\ -x<y<x$$

(8) The joint p.d.f. of (X,Y) is given by $f(x,y)=e^{-(x+y)},\ 0\le x,y<\infty$. Are X & Y independent? Why?

Sol: The marginal density fun/. of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^{\infty} e^{-(x+y)}\,dy=e^{-x}\int_0^{\infty} e^{-y}\,dy=e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty}=e^{-x},\ 0\le x<\infty$$

Marginal density fun/. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^{\infty} e^{-(x+y)}\,dx=e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^{\infty}=e^{-y},\ 0\le y<\infty$$

Consider, $f(x)\cdot f(y)=e^{-x}\cdot e^{-y}=e^{-(x+y)}=f(x,y)$. Hence X & Y are independent.

(9) If the joint p.d.f. of a two-dimensional r.v. (X, Y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0<x<1;\ 0<y<2 \\ 0, & \text{elsewhere} \end{cases}$$

. Find (i) $P(X>\frac{1}{2})$ (ii) $P(Y<X)$ & (iii) $P[Y<\frac{1}{2} / X<\frac{1}{2}]$

Check whether the conditional density funs/. are valid.

Sol: Marginal density fun/. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = \left(x^2 y + \frac{xy^2}{6}\right)_0^2 = 2x^2 + \frac{2x}{3}, \quad 0<x<1$$

Marginal density fun/. of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)_0^1 = \frac{1}{3} + \frac{y}{6}, \quad 0<y<2$$

$$\text{(i) } P(X>\tfrac{1}{2}) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} \left(2x^2 + \frac{2x}{3}\right) dx = \left(\frac{2x^3}{3} + \frac{x^2}{3}\right)_{1/2}^{1}$$

$$= \frac{2}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{12} \cancel{= \frac{16+8-2-1}{24}} \cancel{= \frac{21}{24}} = \frac{8+4-1-1}{12} = \frac{10}{12} = \frac{5}{6}$$

$$\text{(ii) } P(Y<X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\,dx = \int_0^1 \left(x^2 y + \frac{xy^2}{6}\right)_0^x dx = \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx$$

$$= \left(\frac{x^4}{4} + \frac{x^4}{24}\right)_0^1 = \frac{1}{4} + \frac{1}{24} = \frac{6+1}{24} = \frac{7}{24}$$

$$\text{(iii) } P[Y<\tfrac{1}{2} / X<\tfrac{1}{2}] = \frac{P[X<\frac{1}{2}, Y<\frac{1}{2}]}{P[X<\frac{1}{2}]}$$

$$P[X<\tfrac{1}{2}, Y<\tfrac{1}{2}] = \int_0^{1/2} \int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\,dy = \int_0^{1/2} \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)_0^{1/2} dy$$

$$= \int_0^{1/2} \left(\frac{1}{24} + \frac{y}{24}\right) dy = \frac{1}{24} \int_0^{1/2} (1+y)\,dy = \frac{1}{24}\left(y + \frac{y^2}{2}\right)_0^{1/2}$$

$$= \frac{1}{24}\left(\frac{1}{2} + \frac{1}{8}\right) = \frac{1}{24}\left(\frac{4+1}{8}\right) = \frac{5}{192}$$

$$P(X<\tfrac{1}{2}) = \int_0^{1/2} f(x)\,dx = \int_0^{1/2} \left(2x^2 + \frac{2x}{3}\right) dx = \left(\frac{2x^3}{3} + \frac{x^2}{3}\right)_0^{1/2} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\therefore P[Y<\tfrac{1}{2} / X<\tfrac{1}{2}] = \frac{5/192}{1/6} = \frac{5}{192} \times 6 = \frac{5}{32}$$

Checking the conditional density funs/. are valid.

$$\int_0^1 f(x/y)\,dx = \int_0^1 \frac{f(x,y)}{f(y)}\,dx = \int_0^1 \left(\frac{x^2 + xy/3}{1/3 + y/6}\right) dx = \int_0^1 \left(\frac{3x^2+xy}{3} \times \frac{6}{2+y}\right) dx$$

$$= \int_0^1 \left(\frac{6x^2 + 2xy}{2+y}\right) dx = \frac{2}{2+y} \int_0^1 (3x^2 + xy)\,dx = \frac{2}{2+y}\left(x^3 + \frac{x^2 y}{2}\right)_0^1$$

$$= \frac{2}{2+y}\left(1 + \frac{y}{2}\right) = \frac{2}{2+y}\left(\frac{2+y}{2}\right) = 1$$

$$\therefore \int_0^2 f(y/x)\,dy = \int_0^2 \frac{f(x,y)}{f(x)}\,dy = \int_0^2 \left(\frac{3x^2+xy}{3} \times \frac{3}{6x^2+2x}\right) dy$$

$$= \frac{1}{6x^2+2x}\int_0^2 (3x^2+xy)\,dy = \frac{1}{6x^2+2x}\left(3x^2y + \frac{xy^2}{2}\right)_0^2$$

$$= \frac{1}{6x^2+2x}\left(6x^2+2x\right) = 1$$

(10) If the joint p.d.f. of a two-dimensional r.v. (X,Y) is given by

$$f(x,y) = \begin{cases} K(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

. Find (i) the value of K (ii) $P(X<1, Y<3)$ (iii) $P(X+Y<3)$ & (iv) $P(X<1 / Y<3)$.

Sol: (i) WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_2^4\int_0^2 K(6-x-y)\,dx\,dy = 1$

$$\Rightarrow K\int_2^4 \left(6x - \frac{x^2}{2} - xy\right)_0^2 dy = 1 \Rightarrow K\int_2^4 (12-2-2y)\,dy = 1$$

$$\Rightarrow K\int_2^4 (10-2y)\,dy = 1 \Rightarrow 2K\int_2^4 (5-y)\,dy = 1 \Rightarrow 2K\left(5y - \frac{y^2}{2}\right)_2^4 = 1$$

$$\Rightarrow 2K(20-8-10+2) = 1 \Rightarrow 8K = 1 \Rightarrow K = 1/8$$

(ii) $P(X<1, Y<3) = \int_0^1\int_2^3 f(x,y)\,dy\,dx = \frac{1}{8}\int_0^1\int_2^3 (6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^1 \left(6y - xy - \frac{y^2}{2}\right)_2^3 dx = \frac{1}{8}\int_0^1 \left(18 - 3x - \frac{9}{2} - 12 + 2x + 2\right) dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right) dx = \frac{1}{8}\left(\frac{7x}{2} - \frac{x^2}{2}\right)_0^1 = \frac{1}{16}(7-1) = \frac{3}{8}$$

(iii) $P(X+Y<3) = \int_2^3\int_0^{3-y} f(x,y)\,dx\,dy = \frac{1}{8}\int_2^3\int_0^{3-y} (6-x-y)\,dx\,dy$

$P(X+Y<3) = \int_0^1\int_2^{3-x} f(x,y)\,dy\,dx$

$\therefore$ Min. value for y is 2
$x+y<3$ means x value is 1.

$$= \frac{1}{8}\int_2^3 \left(6x - \frac{x^2}{2} - xy\right)_0^{3-y} dy = \frac{1}{8}\int_2^3 \left(18 - 6y - \frac{(3-y)^2}{2} - (3-y)y\right) dy$$

$$= \frac{1}{8}\int_2^3 \left(18 - 6y - \frac{1}{2}(3-y)^2 - 3y + y^2\right) dy = \frac{1}{8}\int_2^3 \left(18 - 9y + y^2 - \frac{1}{2}(3-y)^2\right) dy$$

$$= \frac{1}{8}\left[18y - \frac{9y^2}{2} + \frac{y^3}{3} - \frac{1}{2}\frac{(3-y)^3}{3(-1)}\right]_2^3$$

$$= \frac{1}{8}\left[54 - \frac{81}{2} + 9 + 0 - 36 + 18 - \frac{8}{3} - \frac{1}{6}\right] = \frac{1}{8}\left[45 + \frac{-243-16-1}{6}\right]$$

$$= \frac{1}{8}\left[45 - \frac{130}{3}\right] = \frac{1}{8} \times \frac{5}{3} = \frac{5}{24}$$

(iv) $P(X<1 / Y<3) = \dfrac{P(X<1, Y<3)}{P(Y<3)}$

$$P(Y<3) = \int_2^3 f(y)\,dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx = \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)_0^2$$

$$= \frac{1}{8}(12-2-2y) = \frac{1}{8}(10-2y) = \frac{1}{4}(5-y), \quad 2<y<4$$

$$\therefore P(Y<3) = \int_2^3 \frac{1}{4}(5-y)\,dy = \frac{1}{4}\left(5y - \frac{y^2}{2}\right)_2^3 = \frac{1}{4}\left(15 - \frac{9}{2} - 10 + 2\right) = \frac{1}{4}\times\frac{5}{2} = \frac{5}{8}$$

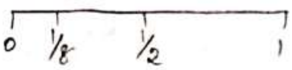
$$\therefore P(X<1 / Y<3) = \frac{3/8}{5/8} = \frac{3}{5}$$

(11) The joint density funf. of the rvs X & Y is given by $f(x,y) = \begin{cases} 8xy, & 0<x<1;\ 0<y<x \\ 0, & \text{elsewhere} \end{cases}$. Find $P(Y<1/8 / X<1/2)$. Also find the conditional density funf. $f(y/x)$.

Sol: $P(Y<1/8 / X<1/2) = \dfrac{P(X<1/2,\ Y<1/8)}{P(X<1/2)}$

$$P(X<1/2,\ Y<1/8) = \int_0^{1/2}\int_0^{x} f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^{x} 8xy\,dy\,dx = 8\int_0^{1/2} x\left(\frac{y^2}{2}\right)_0^x dx$$

$$= \frac{8}{2}\int_0^{1/2} x(x^2)\,dx = 4\int_0^{1/2} x^3\,dx = 4\left(\frac{x^4}{4}\right)_0^{1/2} = \frac{1}{16}$$



$$P(X<1/2) = \int_0^{1/2} f(x)\,dx$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_0^x = 4x(x^2) = 4x^3, \quad 0<x<1$$

$$\therefore P(X<1/2) = \int_0^{1/2} 4x^3\,dx = 4\left(\frac{x^4}{4}\right)_0^{1/2} = \frac{1}{16}$$

$$\therefore P(Y<1/8 / X<1/2) = \frac{1/16}{1/16} = 1$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}, \quad 0<x<1$$

(12) If the joint density funf. of the two rvs X & Y be $f(x,y) = \begin{cases} e^{-(x+y)}, & x\ge 0, y\ge 0 \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(X<1)$ & (ii) $P(X+Y<1)$.

Sol: Marginal density funf. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-x}.e^{-y}\,dy = e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}, \quad x\ge 0$$

$$\text{(i)}\ P(X<1) = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx = \left(\frac{e^{-x}}{-1}\right)_0^1 = -(e^{-1}-1) = 1-e^{-1}$$

$$\text{(ii)}\ P(X+Y<1) = \int_0^1\int_0^{1-y} f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} e^{-x}.e^{-y}\,dx\,dy = \int_0^1 e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^{1-y} dy$$

$$= -\int_0^1 e^{-y}\left(e^{-(1-y)}-1\right)dy = \int_0^1 e^{-y}\left(1-e^{-1+y}\right)dy$$

$$= \int_0^1 \left(e^{-y}-e^{-1}\right)dy = \left[\frac{e^{-y}}{-1} - e^{-1}y\right]_0^1 = \left[-e^{-y}-e^{-1}y\right]_0^1$$

$$= \left[-e^{-1}-e^{-1}+1\right] = 1-2e^{-1}$$

(13) X & Y are two rvs having joint density func. $f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$

Find $P(X<1 \cap Y<3)$.

Sol: $P(X<1 \cap Y<3) = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^1 \left(6y-xy-\frac{y^2}{2}\right)_2^3 dx = \frac{1}{8}\int_0^1 \left(18-3x-\frac{9}{2}-12+2x+2\right)dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{7}{2}-x\right)dx = \frac{1}{8}\left(\frac{7x}{2}-\frac{x^2}{2}\right)_0^1 = \frac{1}{16}(7-1) = \frac{3}{8}$$

(14) Given the joint p.d.f. of (X,Y) as $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$. Find the marginal & conditional pdfs of X & Y. Are X & Y independent?

Sol: Marginal density fun. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_x^1 = 4x(1-x^2), \quad 0<x<1$$

Marginal density fun. of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_0^y = 4y(y^2) = 4y^3, \quad 0<y<1$$

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}, \quad 0<x<y$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}, \quad x<y<1$$

$$f(x)\cdot f(y) = 4x(1-x^2)\cdot 4y^3 = 16xy^3(1-x^2) \neq f(x,y)$$

Hence X & Y are not independent.

Covariance:

(i) $Cov(X,Y) = E(XY) - E(X)E(Y)$

(ii) $Cov(aX, bY) = ab\,cov(X,Y)$

(iii) $Cov(X+a, Y+b) = Cov(X,Y)$

(iv) $Cov(aX+b, cY+d) = ac\,Cov(X,Y)$

(v) $V(X_1+X_2) = V(X_1)+V(X_2)+2\,Cov(X_1,X_2)$

(vi) $V(X_1-X_2) = V(X_1)+V(X_2) - 2\,Cov(X_1,X_2)$

Note: If X & Y are independent, then $Cov(X,Y)=0$

($\because$ X & Y are independent, $E(XY) = E(X)\cdot E(Y)$)

Karl Pearson's coefficient of correlation:

Let X & Y be given random variables. The Karl Pearson's coefficient of correlation is denoted by r_{XY} or $r(X,Y)$ & defined as

$$r(X,Y) = r_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

where $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{\Sigma xy}{n} - \bar{X}\bar{Y}$. Here $\bar{X} = \frac{\Sigma x}{n}$ & $\bar{Y} = \frac{\Sigma y}{n}$

& n is the no. of items in the given data.

$\sigma_X^2 = Var(X) = \frac{1}{n}\Sigma X^2 - \bar{X}^2$ & $\sigma_Y^2 = Var(Y) = \frac{1}{n}\Sigma Y^2 - \bar{Y}^2$

Note: (i) Correlation coefficient always lies between -1 to 1.

(ii) Two rvs with non zero correlation are said to be correlated.

Rank Correlation:

If (x_i, y_i), $i = 1, 2, \ldots, n$ be the ranks of the individuals in two characteristics A & B respectively. Then the rank correlation coefficient is given by

$$r = 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} d_i^2 \quad \text{where } d_i = x_i - y_i \; ; \; n = \text{no. of items.}$$

where d_i is the different between the ranks. This formula is called Spearman's formula for the rank correlation coefficient.

Note: In the correction formula, we add the factor $\frac{m(m^2-1)}{12}$ to Σd^2 where m is the no. of items an item is repeated. This correction factor is to be added for each repeated value.

Problems:

① Calculate the correlation coefficient for the following heights (in inches) of fathers X their sons Y.

X : 65 66 67 67 68 69 70 72

Y : 67 68 65 68 72 72 69 71

Sol:.

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041

Here $\Sigma X = 544$, $\Sigma Y = 552$

$\Sigma XY = 37560$

$\Sigma X^2 = 37028$

$\Sigma Y^2 = 38132$

$E(X) = 68$, $E(Y) = 69$, $E(XY) = 4695$

$E(X^2) = 4628.5$, $E(Y^2) = 4766.5$

$Cov(X,Y) = 3$, $Var(X) = 4.5$,

$Var(Y) = 5.5$

$r(X,Y) = 0.603$

$\bar{X} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$; $\bar{Y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$

$\bar{X}\bar{Y} = 68 \times 69 = 4692$

$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} = \sqrt{\frac{1}{8}(37028) - (68)^2} = \sqrt{4628.5 - 4624} = 2.1213$

$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2} = \sqrt{\frac{1}{8}(38132) - (69)^2} = \sqrt{4766.5 - 4761} = 2.3452$

$Cov(x,y) = \frac{1}{n}\Sigma xy - \bar{X}\bar{Y} = \frac{1}{8}(37560) - (68 \times 69) = 4695 - 4692 = 3$

The correlation coefficient of X & Y is given by

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{3}{(2.1213)(2.3452)} = \frac{3}{4.9749} = 0.603$$

② Two rvs X & Y have the joint density $f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$. Show that $Cor(x,y) = \frac{-1}{11}$.

Sol: Marginal density funᶜ. of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-x-y)\,dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1 = 2 - x - \frac{1}{2} = \frac{3}{2} - x, \quad 0<x<1$$

Marginal density funᶜ. of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (2-x-y)\,dx = \left(2x - \frac{x^2}{2} - xy\right)_0^1 = 2 - \frac{1}{2} - y = \frac{3}{2} - y, \quad 0<y<1$$

Now, $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3}{2}x - x^2\right)dx$

$$= \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

Similarly, $E(Y) = \frac{5}{12}$

$\cancel{E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x - y\right)dx = \int_0^1 (2x^2 - x^3 - x^2 y)\,dx}$

$\cancel{= \left[\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^3 y}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{4} - \frac{y}{3}}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3x^2}{2} - x^3\right)dx$$

$$= \left[\frac{3x^3}{6} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4}$$

Similarly, $E(Y^2) = \frac{1}{4}$

$\therefore \sigma_x^2 = Var(x) = E(x^2) - [E(x)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}$

$\Rightarrow \sigma_x = \frac{\sqrt{11}}{12}$

Similarly $\sigma_y^2 = Var(Y) = \frac{11}{144} \Rightarrow \sigma_y = \frac{\sqrt{11}}{12}$

$$E(XY) = \int_0^1\int_0^1 xy\,f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dx\,dy = \int_0^1 \left(x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2} \right)_0^1 dy$$

$$= \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2} \right) dy = \left[\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{3-2}{6} = \frac{1}{6}$$

$$Cor(X,Y) = E(XY) - E(X).E(Y) = \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144} = \frac{-1}{144}$$

The correlation coefficient is

$$r(X,Y) = \frac{Cor(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{-1}{11}$$

(3) The joint probability mass fun/. of X & Y is given below.

y \ x	-1	1
0	1/8	3/8
1	2/8	2/8

Find correlation coefficient of (X, Y).

Sol:

y \ x	-1	1	$p(y) = P_{\cdot j}$
0	1/8	3/8	$P(Y=0) = 1/2$
1	2/8	2/8	$P(Y=1) = 1/2$
$p(x) = P_{i\cdot}$	$P(X=-1) = 3/8$	$P(X=1) = 5/8$	

$$E(X) = \sum x_i P(x_i) = (-1)\left(\frac{3}{8}\right) + (1)\left(\frac{5}{8}\right) = \frac{-3}{8} + \frac{5}{8} = \frac{2}{8} = \frac{1}{4}$$

$$E(X^2) = \sum x_i^2 P(x_i) = (-1)^2\left(\frac{3}{8}\right) + (1)^2\left(\frac{5}{8}\right) = \frac{3}{8} + \frac{5}{8} = 1$$

$$E(Y) = \sum y_i P(y_i) = (0)(1/2) + 1(1/2) = 1/2$$

$$E(Y^2) = \sum y_i^2 P(y_i) = (0)^2(1/2) + (1)^2(1/2) = 1/2$$

$$E(XY) = \sum_i \sum_j x_i y_j P(x_i, y_j) = (0)(-1)\left(\frac{1}{8}\right) + (0)(1)\left(\frac{3}{8}\right) + (1)(-1)\left(\frac{2}{8}\right) + (1)(1)\left(\frac{2}{8}\right)$$

$$= \frac{-1}{4} + \frac{1}{4} = 0$$

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2 = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\therefore \sigma_X = \frac{\sqrt{15}}{4}$$

$$\sigma_Y^2 = Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \sigma_Y = \frac{1}{2}$$

$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X\sigma_Y} = \frac{0 - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\frac{\sqrt{15}}{4}\times\frac{1}{2}} = \frac{\frac{-1}{8}}{\frac{\sqrt{15}}{8}} = \frac{-1}{\sqrt{15}} = -0.2582$$

④ Suppose that the 2 dimensional rvs (X,Y) has the joint p.d.f.
$f(x,y)=\begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$. Obtain the correlation coefficient between X & Y.

45, 54, 61 → 2

Sol: Marginal density fun. of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left(xy+\frac{y^2}{2}\right)_0^1 = x+\frac{1}{2} \quad , 0<x<1$$

Marginal density fun. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2}+xy\right)_0^1 = \frac{1}{2}+y \quad , 0<y<1$$

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^2+\frac{x}{2}\right)dx = \left(\frac{x^3}{3}+\frac{x^2}{4}\right)_0^1$$

$$= \frac{1}{3}+\frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

Similarly, $E(Y)=\frac{7}{12}$

$$E(X^2)=\int_0^1 x^2 f(x)\,dx = \int_0^1 x^2(x+\tfrac{1}{2})\,dx = \int_0^1 \left(x^3+\frac{x^2}{2}\right)dx = \left(\frac{x^4}{4}+\frac{x^3}{6}\right)_0^1$$

$$= \frac{1}{4}+\frac{1}{6} = \frac{10}{24} = \frac{5}{12}$$

Similarly, $E(Y^2)=\frac{5}{12}$

$$E(XY)=\int_0^1\int_0^1 xy(x+y)\,dx\,dy = \int_0^1\int_0^1 (x^2y+xy^2)\,dx\,dy = \int_0^1 \left(\frac{x^3y}{3}+\frac{x^2y^2}{2}\right)_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \left(\frac{y^2}{6}+\frac{y^3}{6}\right)_0^1 = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

$$Var(X)=E(X^2)-[E(X)]^2 = \frac{5}{12}-\left(\frac{7}{12}\right)^2 = \frac{5}{12}-\frac{49}{144} = \frac{60-49}{144} = \frac{11}{144}$$

$\therefore \sigma_X = \frac{\sqrt{11}}{12}$. Similarly $\sigma_Y = \frac{\sqrt{11}}{12}$

$$Cov(X,Y)=E(XY)-E(X)E(Y) = \frac{1}{3}-\left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{1}{3}-\frac{49}{144} = \frac{48-49}{144} = \frac{-1}{144}$$

$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X\cdot\sigma_Y} = \frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{-1/144}{11/144} = \frac{-1}{11} = -0.0909$$

⑤ Two independent random variables X & Y are defined by, $f(x)=\begin{cases} 4ax, & 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$
$f(y)=\begin{cases} 4by, & 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$. Show that $U=X+Y$ & $V=X-Y$ are uncorrelated.

Sol: Given $f(x)=\begin{cases} 4ax, & 0\le x\le 1 \\ 0, & \text{otherwise} \end{cases}$

$\because f(x)$ is the density fun. of X, $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 4ax\,dx = 1$

$$\Rightarrow 4a\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow 4a\left(\frac{1}{2}\right)=1 \Rightarrow 2a=1 \Rightarrow a=\frac{1}{2}$$

$f(y) = \begin{cases} 4by, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

$\because f(y)$ is the density funt. of Y, $\int_0^1 f(y)dy = 1 \Rightarrow \int_0^1 4by\,dy = 1$

$$\Rightarrow 4b\left(\frac{y^2}{2}\right)_0^1 = 1 \Rightarrow 4b\left(\frac{1}{2}\right) = 1 \Rightarrow 2b = 1 \Rightarrow b = \frac{1}{2}$$

To prove $U = X+Y$ & $V = X-Y$ are uncorrelated. (ie) to prove $Cov(U,V) = 0$.

Let X_1 & X_2 be 2 independent rvs with means 5 & 10 & S.Ds 2 & 3 resp. Obtain r_{UV} where $U = 3X_1 + 4X_2$ & $V = 3X_1 - X_2$.

$Cov(U,V) = E(UV) - E(U)E(V)$

$E(U) = E(X+Y) = E(X) + E(Y)$

$E(V) = E(X-Y) = E(X) - E(Y)$

$E(UV) = E[(X+Y)(X-Y)] = E(X^2 - Y^2) = E(X^2) - E(Y^2)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x(2x)dx = 2\int_0^1 x^2 dx = 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)dy = \int_0^1 y(2y)dy = 2\int_0^1 y^2 dy = 2\left(\frac{y^3}{3}\right)_0^1 = \frac{2}{3}$$

~~$[E(XY) = E(X)E(Y)$ ($\because X$ & Y are independent)~~

~~$= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}]$~~

$E(U) = E(X) + E(Y) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

$E(V) = E(X) - E(Y) = \frac{2}{3} - \frac{2}{3} = 0$

$E(UV) = E(X^2) - E(Y^2) = \frac{1}{2} - \frac{1}{2} = 0$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2(2x)dx = \int_0^1 2x^3 dx = 2\left(\frac{x^4}{4}\right)_0^1 = \frac{1}{2}$$

Similarly, $E(Y^2) = \frac{1}{2}$

$\therefore Cov(U,V) = E(UV) - E(U)E(V)$

$= 0 - \frac{4}{3}(0) = 0$

Hence U & V are uncorrelated.

Regression:

Regression is a mathematical measure of the average relationship between two or more variables interms of the original limits of the data.

Lines of regression:

(i) The line of regression of y on x is given by $y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$. — (1)

(ii) The line of regression of x on y is given by $x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$. — (2)

Regression coefficients:

(i) Regression coefficient of y on x is $r\frac{\sigma_y}{\sigma_x} = b_{yx}$

(ii) Regression coefficient of x on y is $r\frac{\sigma_x}{\sigma_y} = b_{xy}$

Correlation coefficient $r = \pm\sqrt{b_{yx}\, b_{xy}}$

where $b_{yx} = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$; $b_{xy} = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$

Properties of regression lines:

(i) The regression lines pass through $(\bar{x}, \bar{y})$. So $(\bar{x}, \bar{y})$ is the point of intersection of the regression lines.

(ii) When $r = 1$, that is when there is a perfect +ve correlation or when $r = -1$, that is when there is a perfect -ve correlation the eqn/. ① & ② becomes one are the same & so the regression lines coincide.

(iii) When $r = 0$ the eqns/. of the lines are $y = \bar{y}$ & $x = \bar{x}$ which represent perpendicular lines which are parallel to the axis.

(iv) The slopes of the lines are $r\frac{\sigma_y}{\sigma_x}$, $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$. Since the S.D's σ_x & σ_y are +ve, both the slopes are +ve if r is +ve & -ve if r is -ve. That is all the three, namely the two slopes & r are of same sign.

Angle between the regression lines:

The slopes of the regression lines are $m_1 = r\frac{\sigma_y}{\sigma_x}$, $m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$

If θ is the angle between the lines, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_y}{\sigma_x}\;\frac{\frac{1}{r} - r}{1 + \left(\frac{\sigma_y}{\sigma_x}\right)^2} = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1}{r} - r\right)$$

$$\Rightarrow \tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1 - r^2}{r}\right)$$

Note: (i) When $r = 0$, that is, when there is no correlation between x & y.
$\tan\theta = \infty$ (or) $\theta = \frac{\pi}{2}$ & so the regression lines are perpendicular.

(ii) When $r = 1$ or -1, that is, when their is a perfect correlation, +ve or -ve, $\theta = 0$ & so the lines coincide.

Correlation coefficient is the geometric mean between the two regression coefficients:

Proof: WKT $b_{xy} = r\frac{\sigma_x}{\sigma_y}$ & $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$$\Rightarrow (b_{xy})(b_{yx}) = r^2 \frac{\sigma_x}{\sigma_y}\cdot\frac{\sigma_y}{\sigma_x} = r^2$$

$$\Rightarrow r = \pm\sqrt{(b_{xy})(b_{yx})}$$

If one of the regression coefficient is greater than unity the other must be less than unity:

Proof: WKT $r^2 = b_{xy} b_{yx} \leq 1$ —① $\quad -1 \leq r \leq 1 \Rightarrow r^2 \leq 1$

Assume that $b_{xy} > 1$

We have to prove that $b_{yx} < 1$

Since $b_{xy} > 1$; $\frac{1}{b_{xy}} < 1$

$\therefore (1) \Rightarrow b_{xy} b_{yx} \leq 1$; $b_{yx} \leq \frac{1}{b_{xy}} < 1$ $\therefore b_{yx} < 1$

Distinguish between correlation & regression Analysis:

Correlation	Regression
(1) Correlation means relationship between two variables.	- Regression is a mathematical measure of expressing the average relationship between the two variables.
(2) Correlation need not imply cause & effect relationship between the variables.	- Regression analysis clearly indicates the cause & effect relationship between variables.
(3) Correlation coefficient is symmetric (ie) $r_{xy} = r_{yx}$.	- Regression coefficient is not symmetric. (ie) $b_{xy} \neq b_{yx}$
(4) Correlation coefficient is a measure of the direction & degree of linear (length) relationship between two variables.	- Using the relationship between two variables we can predict the dependent variable value for any given independent variable value.

Standard errors of estimate:

The standard error of estimate of x is $S_x = \sigma_x \sqrt{1-r^2}$

The standard error of estimate of y is $S_y = \sigma_y \sqrt{1-r^2}$.

Correlation of grouped data:

When the no/. of observations is large & the variables are grouped, the data can be classified into two way frequency distribution called a correlation table. If there are 'n' classes for X & 'm' classes for Y, there will be (mxn) cells in the two-way table.

The formula for calculating the coefficient of correlation is $r = \frac{P}{\sigma_x \sigma_y}$

where $P = \frac{\Sigma XY f_{XY}}{N} - \left(\frac{\Sigma X f_X}{N}\right)\left(\frac{\Sigma Y f_Y}{N}\right)$

$$\sigma_X^2 = \frac{\Sigma X^2 f_X}{N} - \left(\frac{\Sigma X f_X}{N}\right)^2 \quad \& \quad \sigma_Y^2 = \frac{\Sigma Y^2 f_Y}{N} - \left(\frac{\Sigma Y f_Y}{N}\right)^2$$

Probable error of correlation coefficient:

The probable error of correlation coefficient is given by

$$P.E.(r) = 0.6745 \times S.E.$$

where S.E. is the standard error & is $S.E.(r) = \frac{1-r^2}{\sqrt{n}}$, where r is the correlation coefficient & n is the no. of observation. Thus

$$P.E.(r) = 0.6745\left(\frac{1-r^2}{\sqrt{n}}\right)$$

The reason for taking the factor 0.6745 is that in a normal distribution, the range $\mu = \pm 0.6745$ covers 50% of the total area. This error enables us to find the limits within which correlation coefficient can be expected to vary.

Problems:

① From the following data, find (i) the two regression eqns., (ii) the coefficient of correlation between the marks in Economics & Statistics, (iii) the most likely marks in Statistics when marks in Economics are 30.

Marks in Eco. x :	25	28	35	32	31	36	29	38	34	32
Statistics y :	43	46	49	41	36	32	31	30	33	39

Sol:

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	−7	5	49	25	−35
28	46	−4	8	16	64	−32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	−1	−2	1	4	2
36	32	4	−6	16	36	−24
29	31	−3	−7	9	49	21
38	30	6	−8	36	64	−48
34	33	2	−5	4	25	−10
32	39	0	1	0	1	0
320	380	0	0	140	398	−93

Here $\bar{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$; $\bar{y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{-93}{140} = -0.6643$$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{-93}{398} = -0.2337$$

Eqn. of the line of regression of x on y is

$$x-\bar{x} = b_{xy}(y-\bar{y}) \Rightarrow x - 32 = -0.2337(y-38) \Rightarrow x = -0.2337y + 8.8806 + 32$$

$\Rightarrow x = -0.2337y + 40.8806$

Eqn. of the line of regression of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x}) \Rightarrow y - 38 = -0.6643(x - 32)$

$\Rightarrow y = -0.6643x + \cancel{32.6643}\,21.2576 + 38$

$\Rightarrow y = -0.6643x + \cancel{70.6643}\ 59.2576$

Coefficient of correlation

$r^2 = b_{yx} b_{xy} = (-0.6643)(-0.2337) = 0.1552$

Since both regression coeff. are -ve, r must be -ve

$\Rightarrow r = \pm 0.394 \Rightarrow r = -0.394$

Now we have to find the most likely marks in statistics (y) when marks in Economics (x) are 30.

(ii) $y = -0.6643x + 59.2576$

When $x = 30 \Rightarrow y = 39$

---

② If the eqns. of the two lines of regression of y on x & x on y are respectively, $7x - 16y + 9 = 0$; $5y - 4x - 3 = 0$, calculate the coefficient of correlation, $\bar{x}$ & $\bar{y}$.

Sol: Since both the regression lines pass through $(\bar{x}, \bar{y})$, we get

$7\bar{x} - 16\bar{y} + 9 = 0$ —① $\quad 5\bar{y} - 4\bar{x} - 3 = 0$ —②

① × 4 $\Rightarrow 28\bar{x} - 64\bar{y} + 36 = 0$

② × 7 $\Rightarrow -28\bar{x} + 35\bar{y} - 21 = 0$

$-29\bar{y} + 15 = 0 \Rightarrow \bar{y} = \frac{15}{29}$

Subst. $\bar{y}$ value in ②, we get

$5\left(\frac{15}{29}\right) - 4\bar{x} - 3 = 0 \Rightarrow 4\bar{x} = \frac{-12}{29} \Rightarrow \bar{x} = \frac{-3}{29}$

∴ The mean values of x & y are $-\frac{3}{29}$ & $\frac{15}{29}$.

[$y - \bar{y} = b_{yx}(x - \bar{x})$
$y = b_{yx}x - b_{yx}\bar{x} + \bar{y}$
$\Rightarrow b_{yx} =$ coeff. of x
$y - \frac{15}{29} = \frac{7}{16}x + \frac{7}{16} \times \frac{3}{29}$
$\Rightarrow y = \frac{7}{16}x + \frac{9}{16}$]

The regression eqn. of y on x is,

$7x - 16y + 9 = 0 \Rightarrow 16y = +7x + 9 \Rightarrow y = \frac{7}{16}x + \frac{9}{16}$

$\therefore b_{yx} = \frac{7}{16}$

Similarly, the regression eqn. of x on y is,

$5y - 4x - 3 = 0 \Rightarrow 4x = 5y - 3 \Rightarrow x = \frac{5}{4}y - \frac{3}{4}$

$\therefore b_{xy} = \frac{5}{4}$

Hence the correlation coefficient between X & Y is given by

$r = \pm\sqrt{b_{xy} \times b_{yx}} = \pm\sqrt{\frac{5}{4} \times \frac{7}{16}} = \pm\sqrt{\frac{35}{64}} = \pm 0.7395$

Since both the regressive coefficients are +ve, r must be +ve. $\therefore r = 0.7395$

$$r = \frac{P}{\sigma_x \sigma_y} = \frac{0.164}{(0.766)(0.7384)} = 0.29$$

The regression eqn/. of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \Rightarrow y - 40 = (0.29)\frac{0.7384}{0.766}(x - 60)$$

$$\Rightarrow y - 40 = 0.2796(x - 60) \Rightarrow y = 0.2796x - 16.776 + 40$$

$$\Rightarrow y = 0.2796x + 23.224$$

The regression eqn/. x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \Rightarrow x - 60 = (0.29)\frac{0.766}{0.7384}(y - 40)$$

$$\Rightarrow x - 60 = 0.3008(y - 40) \Rightarrow x = 0.3008y - 12.032 + 60$$

$$\Rightarrow x = 0.3008y + 47.968$$

(4) For the following data find the most likely price at Madras corresponding to the price 70 at Bombay & that at Bombay corresponding to the price 68 at Madras.

	Madras	Bombay
Average price	65	67
S.D. of price	0.5	3.5

S.D. of the difference between the price at Madras & Bombay is 3.1

Sol: Let x denotes the price at Madras & y denotes the price at Bombay.

Given $\bar{x} = 65$; $\bar{y} = 67$, $\sigma_x = 0.5$, $\sigma_y = 3.5$, $\sigma_{x-y} = 3.1$

The correlation coefficient r is given by

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{(0.5)^2 + (3.5)^2 - (3.1)^2}{2(0.5)(3.5)} = \frac{2.89}{3.5} = 0.8257$$

The line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \Rightarrow y - 67 = (0.8257)\frac{3.5}{0.5}(x - 65)$$

$$\Rightarrow y - 67 = 5.7799(x - 65) \Rightarrow y = 5.7799x - 375.6935 + 68$$

$$\Rightarrow y = 5.7799x - 307.6935$$

Put $x = 68$

Then $y = 5.7799(68) - 307.6935 = 85.3397$

$\therefore$ Corresponding to the price 68 at Madras, the most likely price at Bombay is 85.34.

Similarly, the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \Rightarrow x - 65 = (0.8257)\frac{0.5}{3.5}(y - 67)$$

Put $y = 70$, then $x - 65 = (0.8257)\frac{0.5}{3.5}(70 - 67) \Rightarrow x = 65.3539$

$\therefore$ Corresponding to the price 70 at Bombay, the most likely price at Madras is 65.35.

Transformation of Random Variables:

Two funs/. of two random variables:

If (X,Y) is a 2-dimensional random variable with joint p.d.f. $f_{XY}(x,y)$ & if $Z=g(X,Y)$ & $W=h(X,Y)$ are two other rvs then the joint p.d.f. of (Z,W) is given by, $f_{ZW}(z,w)=\dfrac{f_{XY}(x,y)}{|J|}$ where $J=\begin{vmatrix}\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{vmatrix}$

Note: This result holds good, only if the eqn/. $Z=g(X,Y)$ & $W=h(X,Y)$ when solved, give unique values of x & y in terms of z & w.

One fun/. of two random variables:

If a rv Z is defined as $Z=g(X,Y)$, where X & Y are given rvs with joint p.d.f. $f(x,y)$. To find the p.d.f. of Z, we introduce a second random variable $W=h(X,Y)$ & obtain the joint p.d.f. of (Z,W), by using the previous result. Let it be $f_{ZW}(z,w)$. The required p.d.f. of Z is then obtained as the marginal p.d.f. is $f_Z(z)$ is obtained by simply integrating $f_{ZW}(z,w)$ w.r.t. w.

(ie) $f_Z(z)=\int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$

Problems:

① If X & Y are independent RVs with p.d.f. $e^{-x}, x\geq 0$; $e^{-y}, y\geq 0$ respectively. Find the density fun/. of $U=\dfrac{X}{X+Y}$ & $V=X+Y$. Are U & V independent?

Sol: Since X & Y are independent, $f_{XY}(x,y)=e^{-x}.e^{-y}=e^{-(x+y)}, x,y\geq 0$.

Solving the eqn/. $u=\dfrac{x}{x+y}$ & $v=x+y$, we get

$u=\dfrac{x}{v}\Rightarrow uv=x$; $y=v-x=v-uv=v(1-u)$

$x=uv$ & $y=v(1-u)$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix} v & u\\ -v & 1-u\end{vmatrix}=v(1-u)+uv=v-uv+uv=v$$

The joint p.d.f. of (U,V) is given by,

$$f_{UV}(u,v)=|J|\,f_{XY}(x,y)$$
$$=ve^{-(x+y)}=ve^{-(uv+v(1-u))}=ve^{-v} \quad ①$$

The range space of (U,V) is obtained as follows:

$\because x,y\geq 0$, $uv\geq 0$ & $v(1-u)>0$

$\therefore$ Either $u \geq 0$, $v \geq 0$ & $1-u \geq 0 \Rightarrow 1 \geq u$

(ie) $0 \leq u \leq 1$ & $v \geq 0$

(or) $u \leq 0$, $v \leq 0$, $1-u \leq 0$ (ie) $u \leq 0$, $u \geq 1$ which is absurd.

$\therefore$ The range space of (U,V) is given by $0 \leq u \leq 1$ & $v \geq 0$.

$\therefore f_{UV}(u,v) = ve^{-v}$, $0 \leq u \leq 1$ & $v \geq 0$.

The p.d.f. of U is given by, $f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_{0}^{\infty} ve^{-v}\,dv$

$$\Rightarrow f_U(u) = \left[v \cdot \frac{e^{-v}}{-1} - e^{-v} \right]_0^{\infty} = 1$$

(ie) U is uniformly distributed in $(0,1)$.

The p.d.f. of V is given by $f_V(v) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,du = \int_0^1 ve^{-v}\,du$

$$= ve^{-v}(u)_0^1 = ve^{-v}, \; v \geq 0.$$

Now, $f_U(u) \cdot f_V(v) = ve^{-v} = f_{UV}(u,v)$ (by ①)

$\Rightarrow U$ & V are independent rvs.

---

② If X & Y are independent rvs with density funs. $f_X(x) = e^{-x}U(x)$ & $f_Y(y) = 2e^{-2y}U(y)$ $x, y \geq 0$. Find the density fun. of $Z = X+Y$.

Sol: Since X & Y are independent, $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ Note that $U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{o.w} \end{cases}$

$$f_{XY}(x,y) = 2e^{-(x+2y)}, \; x, y \geq 0.$$

Let us consider the auxiliary rv, $W = Y$

$\therefore x = z - w$ & $y = w$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint p.d.f. of (Z,W) is given by,

$$f_{ZW}(z,w) = |J| f_{XY}(x,y) = 2e^{-(x+2y)} = 2e^{-(z-w+2w)} = 2e^{-(z+w)}$$

The range space of (Z,W) is given as follows:

$w \geq 0$, $z - w \geq 0 \Rightarrow z \geq w$, $0 \leq w \leq z$

The p.d.f. of Z is given by,

$$f_Z(z) = \int_0^z f_{ZW}(z,w)\,dw = \int_0^z 2e^{-(z+w)}\,dw = 2e^{-z}\left[\frac{e^{-w}}{-1}\right]_0^z$$

$$= -2e^{-z}(e^{-z} - 1) = 2e^{-z}(1 - e^{-z}) = 2(e^{-z} - e^{-2z}), \; z \geq 0.$$

---

③ The joint p.d.f. of X & Y is given by $f(x,y) = e^{-(x+y)}, x>0, y>0$, find the probability density fun/. of $U = \frac{X+Y}{2}$.

<u>Sol:</u> Given $U = \frac{X+Y}{2}$ (ie) $u = \frac{x+y}{2}$

Let us make the transformation

$u = \frac{1}{2}(x+y)$ & $v = y$ —①

$\Rightarrow u \geq 0$ & $v \geq 0$ ($\because x>0, y>0$)

Also $u \geq v$ (ie) $u \geq 0$ & $0 \leq v \leq u$

From ①, we get, $x = 2u - v$, $y = v$

The jacobian J of the transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$2u - v > 0 \Rightarrow 2u > v$

$v > 0$

$0 < v < 2u \Rightarrow 2u > 0 \Rightarrow u > 0$

The joint p.d.f. of (u,v) is given by

$$f(u,v) = f(x,y)\,|J| = e^{-(x+y)}\cdot 2 = 2e^{-2u}$$

$u \geq 0$, $0 \leq v \leq u$ as $x+y = 2u$

The p.d.f. of u is given by

$$f_U(u) = \int_0^u f(u,v)\,dv = \int_0^u 2e^{-2u}\,dv = 2e^{-2u}(v)_0^u = 2ue^{-2u}, \quad u \geq 0.$$

---

④ If X & Y are independent rvs each normally distributed with mean zero & variance σ^2, find the density funs/. of $R = \sqrt{X^2+Y^2}$ & $\phi = \tan^{-1}\left(\frac{Y}{X}\right)$.

<u>Sol:</u> Since X & Y are independent rvs normally distributed with mean zero & variance σ^2, the joint p.d.f. of X & Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}, \quad -\infty < x, y < \infty.$$

1,3,15,39 → ECE 23/02

Given that $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Since the given transformation is a polar form of (x,y), we have

$x = r\cos\theta$; $y = r\sin\theta$

Hence $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

The joint p.d.f. of (R,ϕ) is given by

$$f_{R\phi}(r,\theta) = f_{XY}(x,y)\,|J| = r\cdot\frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

Since $-\infty < x, y < \infty$, we have $0 \le \theta \le 2\pi$ & $0 \le r < \infty$.

Hence $f_{R\phi}(r,\theta) = \frac{r}{\sigma^2 2\pi} e^{-r^2/2\sigma^2}$, $0 \le \theta \le 2\pi$, $0 \le r < \infty$.

The p.d.f. of R is given by

$$f_R(r) = \int_0^{2\pi} f_{R\phi}(r,\theta)\,d\theta = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \int_0^{2\pi} d\theta = \frac{r}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}} \cdot 2\pi = \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}}, \ 0 \le r < \infty.$$

$$f_\phi(\theta) = \int_0^\infty f_{R\phi}(r,\theta)\,dr = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = \frac{1}{2\pi\sigma^2} \int_0^\infty r e^{-r^2/2\sigma^2} dr$$

Take $u = \frac{r^2}{2\sigma^2} \Rightarrow du = \frac{1}{2\sigma^2} 2r\,dr = \frac{r}{\sigma^2} dr \Rightarrow r\,dr = \sigma^2 du$

$$\therefore f_\phi(\theta) = \frac{1}{2\pi\sigma^2} \int_0^\infty e^{-u} \sigma^2 du = \frac{1}{2\pi} \int_0^\infty e^{-u} du = \frac{1}{2\pi}\left[\frac{e^{-u}}{-1}\right]_0^\infty = \frac{-1}{2\pi}(0-1) = \frac{1}{2\pi}, \ 0 \le \theta \le 2\pi$$

Central Limit Theorem:

Liapounoff's form:

If X_i $(i=1,2,\dots,n)$ be independent random variables such that $E(X_i) = \mu_i$ & $Var(X_i) = \sigma_i^2$ then under certain general conditions, the rv $S_n = X_1 + X_2 + \dots + X_n$ is asympototically normal with mean μ & standard deviation σ where $\mu = \sum_{i=1}^n \mu_i$ & $\sigma^2 = \sum_{i=1}^n \sigma_i^2$ as $n \to \infty$.

Proof: $M_{X_i}(t) = E(e^{tX_i})$

$= e^{t.1} p + e^{t.0} q = q + pe^t$ — (1)

$M_{S_n}(t) = M_{X_1+X_2+\dots+X_n}(t)$

$= M_{X_1}(t) M_{X_2}(t) \dots M_{X_n}(t)$

$= (M_{X_i}(t))^n$ (As X_i's are independent & identically distributed)

$= (q+pe^t) \dots$ n times

$= (q+pe^t)^n$ (by (1)) — (2) which is m.g.f. of a binomial variate with parameters n & p.

Hence by uniqueness thm. of m.g.f.

$S_n \approx B(n,p)$, $B(n,p)$ is the Binomial distribution.

$\therefore E(S_n) = np = \mu$ (Say)

$Var(S_n) = npq = \sigma^2$ (Say)

Let $Z = \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - \mu}{\sigma}$

Then $M_Z(t) = e^{\frac{-\mu t}{\sigma}} M_{S_n}(t/\sigma)$

$$= e^{\frac{-npt}{\sqrt{npq}}}\left(q + pe^{\frac{t}{\sqrt{npq}}}\right)^n \quad (\text{By } ②)$$

$$= \left[1 + \frac{t^2}{2n} + o(n^{-3/2})\right]^n$$

where $o(n^{-3/2})$ represents terms involving $n^{-3/2}$ & higher powers of n in the denominator.

As $n \to \infty$, we get, $\lim_{n\to\infty} M_Z(t) = \lim_{n\to\infty}\left[1 + \frac{t^2}{2n} + o(n^{-3/2})\right]^n$

$= \lim_{n\to\infty}\left(1 + \frac{t^2}{2n}\right)^n = e^{t^2/2}$ which is the m.g.f. of the standard normal variable.

Hence $S_n = X_1 + X_2 + \dots + X_n$ is asymptotically equivalent to $N(\mu, \sigma^2)$ as $n \to \infty$.

Lindberg - Levy's form:

If $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent identically distributed rvs with $E(X_i) = \mu$ & $Var(X_i) = \sigma^2$, $i = 1, 2, \dots$ & if $S_n = X_1 + X_2 + \dots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

Corollary:

If $\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$, then $E(\bar{X}) = \mu$ & $Var(\bar{X}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

$\therefore \bar{X}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$ as $n \to \infty$.

Applications of Central Limit Theorem:

(i) This thm. provides a simple method for computing approximate probabilities of sums of independent random variables.

(ii) It also gives us the wonderful fact that the empirical frequencies of so many natural populations exhibit a bell shaped curve. (ie, a normal curve).

Problems:

① The lifetime of a certain brand of an electric bulb may be considered as a rv with mean 1200h & standard deviation 250h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250h.

Sol: If X_i denotes the lifetime of the light, then we have

Mean $= E(X_i) = 1200 = \mu$; Variance $= Var(X_i) = 250^2 = \sigma^2$

∴ Let us assume that $\bar{x}$ denote the mean lifetime of 60 lights.
By Corollary of Lindeberg-Levy's form of Central limit thm., we have $\bar{x}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.
N(mean, S.D)
(ie) $\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \bar{x}$ follows $N\left(1200, \frac{250}{\sqrt{60}}\right)$

We have to find the probability of the average lifetime of 60 lights exceeds 1250h.
(ie) to find $P(\bar{x} > 1250)$.
$z = \frac{\bar{x} - \mu \rightarrow \text{mean}}{S.D}$ (→ variable)

Let $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$, z a standard normal variable.

$= \frac{\bar{x} - 1200}{250/\sqrt{60}}$

Now, $P(\bar{x} > 1250) = P\left(\frac{\bar{x}-1200}{250/\sqrt{60}} > \frac{1250-1200}{250/\sqrt{60}}\right) = P\left(z > \frac{50 \times \sqrt{60}}{250}\right)$

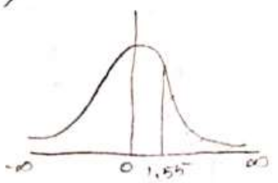
$= P(z > 1.55) = \cancel{P(0<z<\infty) - P(0<z<1.55)}$

$P(0<z<1.55)$

$= 0.5 -$ (Area from 0 to 1.55)

$= 0.5 - 0.4394$ (from the area normal table)

$= 0.0606$



② If $x_1, x_2, \ldots, x_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit thm. to estimate $P(120 \le S_n \le 160)$, where $S_n = x_1 + x_2 + \cdots + x_n$ & $n = 75$.

Sol: Given that $E(x_i) = \lambda = 2 = \mu$ & $Var(x_i) = \lambda = 2 = \sigma^2$

($\because$ For Poisson distribution Mean = Variance = λ], $i = 1, 2, \ldots, n$.

By Central limit thm., we have S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$. (ie) S_n follows $N(n\mu, \sigma\sqrt{n})$.

Also $n = 75$.

Hence S_n follows $N(75 \times 2, \sqrt{2} \times \sqrt{75})$

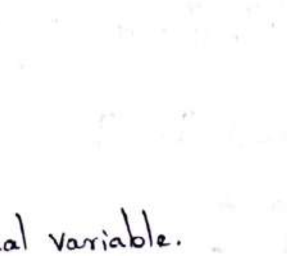
$\Rightarrow S_n$ follows $N(150, \sqrt{150})$

To find $P(120 \le S_n \le 160)$

Let $z = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 150}{\sqrt{150}}$, z is a standard normal variable.

If $S_n = 120$, $z = \frac{120-150}{\sqrt{150}} = \frac{-30}{\sqrt{150}} = -2.45$ — ①

If $S_n = 160$, $z = \frac{160-150}{\sqrt{150}} = \frac{10}{\sqrt{150}} = 0.82$



Now, $P(120 \le S_n \le 160) = P\left(\frac{120-150}{\sqrt{150}} \le z \le \frac{160-150}{\sqrt{150}}\right)$

$= P(-2.45 \le z \le 0.82)$

$= P(-2.45 \le z \le 0) + P(0 \le z \le 0.82)$

$= P(0 \le z \le 2.45) + P(0 \le z \le 0.82)$

$= 0.4929 + 0.2939 = 0.7868$

(3) Let $X_1, X_2, \ldots, X_{100}$ be independent identically distributed random variables with $\mu = 2$ & $\sigma^2 = \frac{1}{4}$. Find $P(192 < X_1 + X_2 + \cdots + X_{100} < 210)$.

Sol: Given that $E(X_i) = \mu = 2$ & $Var(X_i) = \frac{1}{4} = \sigma^2$, $i = 1, 2, \ldots, 100$.

By Central limit thm., we have S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$, $S_n = X_1 + X_2 + \cdots + X_{100}$.

(ie) S_n follows $N(n\mu, \sigma\sqrt{n})$

Hence S_n follows $N(100 \times 2, \frac{1}{2}\sqrt{100}) \Rightarrow S_n$ follows $N(200, 5)$.

To find $P(192 < S_n < 210)$

Let $Z = \dfrac{S_n - n\mu}{\sigma\sqrt{n}} = \dfrac{S_n - 200}{5}$, Z is a standard normal variable.

If $S_n = 192$, $Z = \dfrac{192 - 200}{5} = \dfrac{-8}{5} = -1.6$ —— (1)

If $S_n = 210$, $Z = \dfrac{210 - 200}{5} = \dfrac{10}{5} = 2$ —— (2)

Now, $P(192 < S_n < 210) = P\left(\dfrac{\overset{192}{S_n} - 200}{5} < Z < \dfrac{\overset{210}{S_n} + 200}{5}\right)$

$= P(-1.6 < Z < 2)$ $(\because$ By (1) & (2)$)$

$= P(-1.6 < Z < 0) + P(0 < Z < 2)$

$= P(0 < Z < 1.6) + P(0 < Z < 2)$

$= 0.4452 + 0.4772 = 0.9224$

(4) A random sample of size 100 is taken from a population whose mean is 60 & variance is 400. Using Central limit thm., with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4.

Sample mean $= \bar{x}$

Sol: Given that $n = 100$, $\mu = 60$, $\sigma^2 = 400$.

By the corollary of Lindeberg-Levy form of Central limit thm., $\bar{x}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.

(ie) $\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \bar{x}$ follows $N\left(60, \frac{20}{\sqrt{100}}\right) \Rightarrow N(60, 2)$

To find $P(|\bar{x} - \mu| \le 4)$

Now, $P(|\bar{x} - \mu| \le 4) = P(-4 \le \bar{x} - \mu \le 4) = P(-4 \le \bar{x} - 60 \le 4)$

$= P(56 \le \bar{x} \le 64)$

Let $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-60}{20/\sqrt{100}} = \frac{\bar{x}-60}{2}$

When $\bar{x} = 56$, $z = \frac{56-60}{2} = -2$ — (1)

When $\bar{x} = 64$, $z = \frac{64-60}{2} = 2$ — (2)

Hence, $P(56 \le \bar{x} \le 64) = P\left(\frac{56-60}{2} \le \frac{\bar{x}-60}{2} \le \frac{64-60}{2}\right)$

$= P(-2 \le z \le 2)$ ($\because$ by (1) & (2))

$= P(-2 \le z \le 0) + P(0 \le z \le 2)$

$= P(0 \le z \le 2) + P(0 \le z \le 2) = 2P(0 \le z \le 2)$

$= 2(0.4772) = 0.9544$

(5) A distribution with unknown mean μ has variance equal to 1.5. Use Central limit thm. to determine how large a sample should be taken from the distribution in order that the probability will be atleast 0.95 that the sample mean will be within 0.5 of the population mean.

Sample mean $= \bar{x}$

Population mean $= \mu$

Sol: Let n be the size of the sample.

Given $E(x) = \mu = $ mean, $Var(x) = \sigma^2 = 1.5$

Let $\bar{x}$ be the sample mean. By Corollary of Lindberg-Levy's form of Central limit thm., we have $\bar{x}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.

(ie) $\bar{x}$ follows $N(\mu, \frac{\sigma}{\sqrt{n}}) \Rightarrow \bar{x}$ follows $N(\mu, \frac{\sqrt{1.5}}{\sqrt{n}})$

To find n such that $P(|\bar{x}-\mu| < 0.5) \ge 0.95$.

Consider, $P(|\bar{x}-\mu| < 0.5) \ge 0.95$

$\Rightarrow P(-0.5 < \bar{x}-\mu < 0.5) \ge 0.95$

Let $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-\mu}{\sqrt{1.5}/\sqrt{n}}$, z a standard normal variable.

$\therefore P(-0.5 < \bar{x}-\mu < 0.5) \ge 0.95$

$\Rightarrow P\left(\frac{-0.5\sqrt{n}}{\sqrt{1.5}} < z < \frac{0.5\sqrt{n}}{\sqrt{1.5}}\right) \ge 0.95$

$\Rightarrow P(|z| < 0.4082\sqrt{n}) \ge 0.95$

$P(-0.4082\sqrt{n} < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow P(-0.4082\sqrt{n} < z < 0) + P(0 < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow 2P(0 < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow P(0 < z < 0.4082\sqrt{n}) = 0.475$

The least value of n is obtained from $P(|z| < 0.4082\sqrt{n}) = 0.95$

From the table of areas under normal curve,

$P(|z| < 1.96) = 0.95$

$\Rightarrow 0.4082\sqrt{n} = 1.96$

$\Rightarrow \sqrt{n} = \frac{1.96}{0.4082} \Rightarrow n = \left(\frac{1.96}{0.4082}\right)^2 \Rightarrow n = 23$

$\therefore$ The size of the sample must be atleast 23.

Central limit theorem: (Laplace discovered)

Let $X_1, X_2, \ldots$ be a sequence of independent & identically distributed rvs each having mean μ & variance σ^2. Then the distribution of $\frac{X_1+X_2+\cdots+X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n\to\infty$. That is, for $-\infty < a < \infty$, $P\left\{\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx$ as $n\to\infty$.

Liapounoff's Form:

If $X_1, X_2, \ldots, X_n$ be a sequence of independent rvs with $E(X_i)=\mu_i$ & $Var(X_i) = \sigma_i^2$, $i=1,2,\ldots,n$ & if $S_n = X_1+X_2+\cdots+X_n$ then under certain general conditions, S_n follows a normal distribution with mean $\mu = \sum_{i=1}^{n}\mu_i$ & variance $\sigma^2 = \sum_{i=1}^{n}\sigma_i^2$ as $n\to\infty$.

Lindberg-Levy's Form:

If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed rvs with $E(X_i)=\mu$ & $Var(X_i)=\sigma^2$, $i=1,2,\ldots,n$ & if $S_n = X_1+X_2+\cdots+X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n\to\infty$.

Two Dimensional Random Variables

① Let X & Y be two independent rvs with $Var(X)=9$ & $Var(Y)=3$. Find $Var(4X-2Y+6)$.

Sol: $Var(4X-2Y+6) = 4^2 Var(X) + (-2)^2 Var(Y) = (16\times 9) + (4\times 3) = 156$

② The joint pdf of (X,Y) is $f(x,y) = \begin{cases} 4xy, & 0\le x\le 1, 0\le y\le 1. \\ 0, & \text{otherwise} \end{cases}$ Calculate $P(X\le 2Y)$.

Sol: $P(X\le 2Y) = \int_0^{1/2}\int_0^{2y} 4xy\,dx\,dy$

$$= 4\int_0^{1/2} y\left(\frac{x^2}{2}\right)_0^{2y} dy = 2\int_0^{1/2} y(4y^2)\,dy = 8\left(\frac{y^4}{4}\right)_0^{1/2} = 2\left(\tfrac{1}{16}\right) = \tfrac{1}{8}$$

③ Define covariance & coefficient of correlation between 2 rvs x & y.

Sol: $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$\sigma_X = \sqrt{Var X}$ & $\sigma_Y = \sqrt{Var Y}$, $Var(X) = E(X^2) - [E(X)]^2$, $Var(Y) = E(Y^2) - [E(Y)]^2$

④ The joint pdf of a bivariate rv (X,Y) is given by $f(x,y) = \begin{cases} k, & 0 < y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$ where k is a constant. Determine the value of k.

Sol: WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$k\int_0^1\int_0^x dy\,dx = 1 \Rightarrow k\int_0^1 (y)_0^x dx = 1 \Rightarrow k\int_0^1 x\,dx = 1$$

$$k\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow k\left(\tfrac{1}{2}\right) = 1 \Rightarrow k = 2$$

⑤ P.T. the correlation coeff. ρ_{xy} of the rvs X & Y takes value in the range -1 & 1.

Sol: WKT $r = \dfrac{\rho}{\sigma_X \sigma_Y}$

$$\rho^2 = \left[\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}\right]^2 = \left(\frac{\Sigma XY}{n}\right)^2$$

$$\sigma_X^2 \sigma_Y^2 = \frac{\Sigma X^2 \Sigma Y^2}{n^2}$$

$$\left(\sum (xy)\right)^2 \le \left(\sum x^2\right)\left(\sum y^2\right)$$

$$\frac{\left(\sum xy\right)^2}{n^2} \le \frac{\sum x^2 \sum y^2}{n^2}$$

$$\rho^2 \le \sigma_x^2 \sigma_y^2$$

$$(r\sigma_x\sigma_y)^2 \le \sigma_x^2\sigma_y^2$$

$$r^2\sigma_x^2\sigma_y^2 \le \sigma_x^2\sigma_y^2$$

$$r^2 \le 1 \Rightarrow |r| \le 1 \Rightarrow -1 \le r \le 1$$

---

(6) Let (x,y) be a two-dimensional rv. Define covariance of (x,y). If x & y are independent. What will be the covariance of (x,y)?

Sol: $Cov(x,y) = E(xy) - E(x)E(y)$

$= E(x)E(y) - E(x)E(y)$ ($\because x$ & y are independent)

$= 0$

---

(7) Can $y = 5 + 2.8x$ & $x = 3 - 0.5y$ be the estimated regression eqn. of y on x respectively explain your answer.

Sol: $b_{yx} = 2.8$, $b_{xy} = -0.5$

$r = \pm\sqrt{b_{xy} \cdot b_{yx}}$ = imaginary

$\therefore$ They cannot be estimated regression eqns.

---

(8) The joint pmf of a 2-dimensional rv (x,y) is given by $P(x,y) = k(2x+y)$, $x = 1,2$ & $y = 1,2$, where k is a constant. Find the value of k.

Sol:

$x \backslash y$	1	2
1	3k	4k
2	5k	6k

$18k = 1$

$k = 1/18$

---

(9) If the joint pdf of (x,y) is $f(x,y) = \begin{cases} 1/4, & 0 \le x, y \le 2 \\ 0, & \text{otherwise} \end{cases}$, find $P(x+y \le 1)$.

Sol: $P(x+y \le 1) = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy = \frac{1}{4}\int_0^1\int_0^{1-y} dx\,dy$

$= \frac{1}{4}\int_0^1 (x)_0^{1-y} dy = \frac{1}{4}\int_0^1 (1-y)\,dy = \frac{1}{4}\left(y - \frac{y^2}{2}\right)_0^1$

$= \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8}$

(10) Determine the value of the constant c if the joint density func. of 2 discrete rvs X & Y is given by $p(m,n) = cmn$, $m = 1, 2, 3$ & $n = 1, 2, 3$.

Sol: Given $p(m,n) = cmn$, $m = 1, 2, 3$ & $n = 1, 2, 3$

n\m	1	2	3
1	c	2c	3c
2	2c	4c	6c
3	3c	6c	9c

$36c = 1$

$c = 1/36$

Scanned with CamScanner

ANALYTIC FUNCTIONS

Analytic function: [Holomorphic fun. (or) Regular fun.]

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire function: [Integral fun.]

A function which is analytic everywhere in the finite plane is called an entire function.

The necessary condition for $f(z)$ to be analytic: [Cauchy-Riemann equations]

The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ (ie) $u_x = v_y$ & $v_x = -u_y$.

Sufficient conditions for $f(z)$ to be analytic:

If the partial derivatives u_x, v_x, u_y & v_y are all continuous in D & $u_x = v_y$ & $u_y = -v_x$. Then the function $f(z)$ is analytic in a domain D.

---

① Is $f(z) = z^n$ analytic function everywhere? [N/D 2015]

Sol: Let $z = re^{i\theta}$

$$z^n = r^n (e^{i\theta})^n = r^n e^{in\theta} = r^n[\cos n\theta + i\sin n\theta]$$

$$z^n = r^n \cos n\theta + i r^n \sin n\theta$$

$$u = r^n \cos n\theta \qquad v = r^n \sin n\theta$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta \qquad \frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = r^n(-\sin n\theta . n) = -nr^n \sin n\theta \qquad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

$$\frac{\partial u}{\partial r} = \frac{nr^n}{r}\cos n\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = \frac{nr^n}{r}\sin n\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

C-R eqns. are satisfied & the partial derivative are continuous.

Hence the function is analytic everywhere.

---

② State & prove the necessary conditions for $f(z)$ to be analytic. [N/D - 2015]

Sol: Statement: The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

(ie) $u_x = v_y$ & $v_x = -u_y$.

Proof: Let $f(z)=u(x,y)+iv(x,y)$ be an analytic function at the point z in a region R. Since $f(z)$ is analytic its derivative $f'(z)$ exists in R

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

Let $z=x+iy$

$\Delta z=\Delta x+i\Delta y$

$z+\Delta z=(x+\Delta x)+i(y+\Delta y)$

$f(z)=u(x,y)+iv(x,y)$

$f(z+\Delta z)=u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)$

$f(z+\Delta z)-f(z)=u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)-[u(x,y)+iv(x,y)]$

$=[u(x+\Delta x,y+\Delta y)-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]$

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{[u(x+\Delta x,y+\Delta y)-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]}{\Delta x+i\Delta y}$$

Case(i): If $\Delta z\to 0$ first we assume that $\Delta y=0$ & $\Delta x\to 0$.

$$\therefore f'(z)=\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)-u(x,y)]+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\quad\text{--- (1)}$$

Case(ii): $\Delta z\to 0$, now we assume that $\Delta x=0$ & $\Delta y\to 0$.

$$\therefore f'(z)=\lim_{\Delta y\to 0}\frac{[u(x,y+\Delta y)-u(x,y)]+i[v(x,y+\Delta y)-v(x,y)]}{i\Delta y}$$

$$=\frac{1}{i}\lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}+\lim_{\Delta y\to 0}\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}$$

$$=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\quad\text{--- (2)}$$

From (1) & (2), we get $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$

Equating the real & imaginary parts we get

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\quad \text{(i.e.) } u_x=v_y,\ v_x=-u_y$$

The above equations are known as Cauchy-Riemann equations or C-R equations.

(3) Prove that every analytic function $w = u + iv$ can be expressed as a function of z alone, not as a function of $\bar{z}$. [M/J-2010]

Sol: Let $z = x + iy$ & $\bar{z} = x - iy$

$$x = \frac{z+\bar{z}}{2} \quad \& \quad y = \frac{z-\bar{z}}{2i} \Rightarrow \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$

Hence u & v & also w may be considered as a function of z & $\bar{z}$.

Consider
$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$
$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right) = \frac{u_x}{2} + \frac{i}{2}u_y + \frac{i}{2}v_x - \frac{v_y}{2}$$
$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$
$$= 0 \quad (\because u_x = v_y,\ u_y = -v_x)$$

This means that w is independent of $\bar{z}$. (ie) w is a function of z alone.

Defns.:

Laplace equation in two dimension: $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$

Laplacian operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Laplace equation in 3-dimension: $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$

Laplace equation in polar coordinates: $\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0$.

Harmonic function (or) Potential function:

A real function of two real variables x & y that possesses continuous second order partial derivatives & that satisfies Laplace equation is called a harmonic function.

Conjugate harmonic function: If u & v are harmonic functions such that $u+iv$ is analytic, then each is called the conjugate harmonic function of the other.

(4) If $f(z) = u(x,y) + iv(x,y)$ is an analytic function, show that the curves $u(x,y) = c_1$ & $v(x,y) = c_2$ cut orthogonally. [N/D 2016]

Sol: Given $f(z)$ is an analytic function.

$\therefore$ By C-R equation $u_x = v_y$ & $u_y = -v_x$.

Given: $u(x,y) = c_1$ & $v(x,y) = c_2$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 , \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (Say)} , \quad \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \ (\because \text{by C-R}) = m_2 \text{ (Say)}$$

Product of slopes at their point of intersection $= m_1 m_2 = \dfrac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \dfrac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1$

Hence the two family of curves form an orthogonal system.

(5) If $f = u+iv$ is analytic on a domain D & $|f|$ is a constant on D, prove that f must be a constant on D. (or) An analytic function with constant modulus is constant. [N/D - 2014]

Sol: Let $f(z) = u+iv$ be analytic.

By C-R equations $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$|f(z)| = \sqrt{u^2+v^2} = c \neq 0$$

$$|f(z)|^2 = u^2+v^2 = c^2 \qquad \text{(i.e.) } u^2+v^2 = c^2$$

Diff/. part/. w.r.t. x,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{--- (1)}$$

Diff/. part/. w.r.t. y,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \Rightarrow u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \Rightarrow -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0 \quad \text{--- (2)}$$

$$(1) \times u \Rightarrow u^2\frac{\partial u}{\partial x} + \cancel{uv\frac{\partial v}{\partial x}} = 0$$

$$(2) \times v \Rightarrow v^2\frac{\partial u}{\partial x} - \cancel{uv\frac{\partial v}{\partial x}} = 0$$

$$(u^2+v^2)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \quad (\because u^2+v^2 \neq 0)$$

$$(1) \times v \Rightarrow \cancel{uv\frac{\partial u}{\partial x}} + v^2\frac{\partial v}{\partial x} = 0$$

$$(2) \times u \Rightarrow \cancel{uv\frac{\partial u}{\partial x}} - u^2\frac{\partial v}{\partial x} = 0$$

$$(-) \qquad (+) \qquad (-)$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0 \quad (\because u^2+v^2 \neq 0)$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i0 = 0 \Rightarrow f(z) = c \text{ is constant.}$$

(6) If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

[NOV/DEC - 2014] [M/J - 2009] [A/M - 2011]

Sol: Let $f(z)=u+iv$

Given $f(z)$ is an analytic function. $\Rightarrow u_x = v_y$ & $u_y = -v_x$ by C-R equns.

$$\overline{f(z)} = \overline{u+iv} = u-iv$$

$$f(z)\overline{f(z)} = (u+iv)(u-iv) = u^2-(iv)^2 = u^2+v^2 \Rightarrow |f(z)|^2 = u^2+v^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = \frac{\partial^2}{\partial x^2}(u^2+v^2)+\frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$= \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+\frac{\partial^2 v^2}{\partial y^2} \quad \text{--- (1)}$$

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$

$$\therefore \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right] = 2\left[u(0)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

($\because u$ is harmonic)

$$= 2\left[u_x^2+u_y^2\right] = 2\left[u_x^2+(-v_x)^2\right]$$

$$= 2\left[u_x^2+v_x^2\right] = 2|f'(z)|^2 \quad (\because f'(z) = u_x+iv_x \Rightarrow |f'(z)|^2 = u_x^2+v_x^2)$$

Similarly, $\frac{\partial^2 v^2}{\partial x^2}+\frac{\partial^2 v^2}{\partial y^2} = 2|f'(z)|^2$

$$\therefore (1) \Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2|f'(z)|^2+2|f'(z)|^2 = 4|f'(z)|^2$$

Hence the proof.

(7) If $f(z)=u+iv$ is an analytic function then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p) = p(p-1)(u^{p-2})|f'(z)|^2$. [April/May - 2018]

Sol: Let $f(z)=u+iv$ is an analytic function. $\Rightarrow u_x = v_y$ — (1) & $u_y = -v_x$ — (2) by C-R equns.

$\Rightarrow u_{xx}+u_{yy}=0$ & $v_{xx}+v_{yy}=0$ [$\because u$ & v are harmonic functions]

$\Rightarrow u_x v_x + u_y v_y = 0$ by (1) & (2)

$$f'(z) = u_x+iv_x \Rightarrow |f'(z)| = \sqrt{u_x^2+v_x^2}$$

$$|f'(z)|^2 = u_x^2+v_x^2$$

$$\frac{\partial}{\partial x}(u^p) = p u^{p-1} u_x$$

$$\frac{\partial^2}{\partial x^2}(u^p) = \frac{\partial}{\partial x}(p u^{p-1} u_x) = p\left[u^{p-1} u_{xx} + u_x (p-1) u^{p-2} u_x\right]$$

$$= p\left[u^{p-1} u_{xx} + (p-1) u^{p-2} u_x^2\right]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^p) = p\left[u^{p-1} u_{yy} + (p-1) u^{p-2} u_y^2\right]$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^p) = p u^{p-1}\left[u_{xx} + u_{yy}\right] + p(p-1) u^{p-2}\left[u_x^2 + u_y^2\right]$$

$$= 0 + p(p-1) u^{p-2}\left[u_x^2 + (-v_x)^2\right] \quad (\because u_{xx} + u_{yy} = 0)$$

$$= p(p-1) u^{p-2}\left[u_x^2 + v_x^2\right] = p(p-1) u^{p-2} |f'(z)|^2$$

Hence the proof.

(8) If $f(z) = u + iv$ is an analytic function in $z = x + iy$ then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^2 = 2|f'(z)|^2$. [A/M-2016]

Sol: Let $f(z) = u + iv$ is an analytic function.

$$\text{Re} f(z) = u$$

$$|\text{Re} f(z)|^2 = u^2 = |u|^2$$

$$\frac{\partial}{\partial x}(u^2) = 2u u_x \quad ; \quad \frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2u u_x) = 2\left[u u_{xx} + u_x u_x\right] = 2\left[u u_{xx} + u_x^2\right]$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u u_{yy} + u_y^2\right]$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2\left[u u_{xx} + u_x^2 + u u_{yy} + u_y^2\right]$$

$$= 2\left[u(u_{xx} + u_{yy}) + u_x^2 + (-v_x)^2\right]$$

$$= 2\left[u(0) + u_x^2 + v_x^2\right] = 2\left[u_x^2 + v_x^2\right] = 2|f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^2 = 2|f'(z)|^2$$

Hence the proof.

(9) If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$.

Sol. We know that $|f(z)|^2 = u^2 + v^2$ $(\because f(z) = u + iv)$ [A/M-2017]

$$|f(z)| = (u^2 + v^2)^{1/2}$$

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$$

$$\frac{\partial}{\partial x}\left[\log|f(z)|\right] = \frac{1}{2}\frac{\partial}{\partial x}\left[\log(u^2 + v^2)\right] = \frac{1}{2}\left[\frac{1}{u^2 + v^2}\right]\left[2u u_x + 2v v_x\right] = \frac{u u_x + v v_x}{u^2 + v^2}$$

$$\frac{\partial^2}{\partial x^2}[\log|f(z)|] = \frac{\partial}{\partial x}\left[\frac{uu_x + vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)(uu_{xx} + u_x^2 + vv_{xx} + v_x^2) - (uu_x + vv_x)(2uu_x + 2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)(uu_{xx} + vv_{xx} + u_x^2 + v_x^2) - 2(uu_x + vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{\partial^2}{\partial y^2}[\log|f(z)|] = \frac{(u^2+v^2)(uu_{yy} + vv_{yy} + u_y^2 + v_y^2) - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\log|f(z)|] = \frac{(u^2+v^2)[uu_{xx} + vv_{xx} + u_x^2 + v_x^2 + uu_{yy} + vv_{yy} + u_y^2 + v_y^2] - 2(uu_x + vv_x)^2 - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + v_y^2 + (-v_x)^2] - 2(uu_x + vv_x)^2 - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u_x^2 + v_x^2 + v_y^2 + v_x^2] - 2[u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y]}{(u^2+v^2)^2}$$

$(\because u_x = v_y,\ u_y = -v_x,$
$u_{xx} + u_{yy} = 0\ \&\ v_{xx} + v_{yy} = 0)$

$$= \frac{(u^2+v^2)[u_x^2 + v_x^2 + u_x^2 + v_x^2] - 2[u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) + 2uvu_xv_x + 2uvu_yv_y]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[2(u_x^2 + v_x^2)] - 2[u^2(u_x^2 + v_x^2) + v^2(v_x^2 + u_x^2) + 2uv(u_xv_x + u_yv_y)]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[2(u_x^2 + v_x^2)] - 2[(u_x^2 + v_x^2)(u^2+v^2) + 2uv(u_xv_x - v_xu_x)]}{(u^2+v^2)^2}$$

$$= \frac{2(u^2+v^2)(u_x^2 + v_x^2) - 2(u^2+v^2)(u_x^2 + v_x^2)}{(u^2+v^2)^2} = 0$$

$\therefore \nabla^2 \log|f(z)| = 0$

Hence the proof.

(10) If $f(z)$ is analytic function of z in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f'(z)|^2|f(z)|^{p-2}.$$ [N/D-2011]

Sol: Let $f(z)=u+iv$ is an analytic function. $f'(z)=u_x+iv_x$

$|f(z)|^2=u^2+v^2 \Rightarrow |f(z)|=(u^2+v^2)^{1/2}$ $\qquad |f'(z)|^2=u_x^2+v_x^2$

$|f(z)|^p=(u^2+v^2)^{p/2}$

$$\frac{\partial}{\partial x}|f(z)|^p=\frac{\partial}{\partial x}\left[(u^2+v^2)^{p/2}\right]=\frac{p}{2}(u^2+v^2)^{\frac{p}{2}-1}\left[2uu_x+2vv_x\right]$$

$$=p(u^2+v^2)^{\frac{p}{2}-1}\left[uu_x+vv_x\right]$$

$$\frac{\partial^2}{\partial x^2}|f(z)|^p=\frac{\partial}{\partial x}\left[p(u^2+v^2)^{\frac{p}{2}-1}(uu_x+vv_x)\right]$$

$$=p(u^2+v^2)^{\frac{p}{2}-1}(uu_{xx}+u_x^2+vv_{xx}+v_x^2)+p(uu_x+vv_x)\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}(2uu_x+2vv_x)$$

$$=p(u^2+v^2)^{\frac{p-2}{2}}(uu_{xx}+vv_{xx}+|f'(z)|^2)+p\left(\frac{p-2}{2}\right)(u^2+v^2)^{\frac{p-4}{2}}2(uu_x+vv_x)^2$$

$$=p(u^2+v^2)^{\frac{p-2}{2}}(uu_{xx}+vv_{xx}+|f'(z)|^2)+p(p-2)(u^2+v^2)^{\frac{p-4}{2}}(u^2u_x^2+v^2v_x^2+2uvu_xv_x)$$

Similarly,

$$\frac{\partial^2}{\partial y^2}|f(z)|^p=p(u^2+v^2)^{\frac{p-2}{2}}(uu_{yy}+vv_{yy}+|f'(z)|^2)+p(p-2)(u^2+v^2)^{\frac{p-4}{2}}(u^2u_y^2+v^2v_y^2+2uvu_yv_y)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p=p(u^2+v^2)^{\frac{p-2}{2}}\left[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+2|f'(z)|^2\right]$$
$$+p(p-2)(u^2+v^2)^{\frac{p-4}{2}}\left[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_xv_x+u_yv_y)\right]$$

$$=2p(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2+p(p-2)(u^2+v^2)^{\frac{p-4}{2}}\left[u^2(u_x^2+v_x^2)+v^2(v_x^2+u_x^2)\right]$$

$(\because u_x=v_y \ \& \ u_y=-v_x)$

$$=2p(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2+p(p-2)(u^2+v^2)^{\frac{p-4}{2}}(u^2+v^2)(u_x^2+v_x^2)$$

$$=2p(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2+p(p-2)(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2$$

$$=p(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2\left[2+p-2\right]$$

$$=p^2(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2=p^2\left(|f(z)|^2\right)^{\frac{p-2}{2}}|f'(z)|^2$$

$$=p^2|f(z)|^{p-2}|f'(z)|^2$$

Hence the proof.

(11) Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Find also its conjugate. [A/M - 2011]

Sol: Given $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

Hence u satisfies the Laplace equation. $\therefore u$ is harmonic.

Here $\varphi_1(x,y) = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$ & $\varphi_2(x,y) = \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\varphi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}$, $\varphi_2(z,0) = \frac{0}{z^2} = 0$

By Milne's Thomson method,

$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz = \int \frac{1}{z}\,dz - i\int 0\,dz = \log z + c$

$f(z) = \log z + c$

$x = r\cos\theta$, $y = r\sin\theta$

$x+iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta) = re^{i\theta}$

$u+iv = \log(x+iy) + c = \log(re^{i\theta}) + c$

$= \log r + \log e^{i\theta} + c = \log\sqrt{x^2+y^2} + i\theta + c$

$= \frac{1}{2}\log(x^2+y^2) + i\theta + c = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} + c$

$\therefore v = \tan^{-1}\frac{y}{x}$

(12) Find the analytic function $f(z) = u+iv$ whose real part is $u = e^x(x\cos y - y\sin y)$. Find also the conjugate harmonic of u. [N/D 2016]

Sol: Given $u = e^x(x\cos y - y\sin y) = e^x x\cos y - e^x y\sin y$

$\varphi_1(x,y) = \frac{\partial u}{\partial x} = \cos y\left[e^x(1) + xe^x\right] - e^x y\sin y = e^x\cos y + xe^x\cos y - ye^x\sin y$

$\varphi_1(z,0) = e^z\cos 0 + ze^z\cos 0 - (0)e^z\sin 0 = e^z + ze^z$

$\varphi_2(x,y) = \frac{\partial u}{\partial y} = e^x x(-\sin y) - e^x(y\cos y + \sin y(1)) = -xe^x\sin y - ye^x\cos y - e^x\sin y$

$\varphi_2(z,0) = -ze^z\sin 0 - (0)e^z\cos 0 - e^z\sin 0 = 0$

By Milne's thomson method,

$f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz = \int (e^z + ze^z)dz - i\int 0\,dz = \int e^z(z+1)dz$

$u = z+1, \quad dv = e^z dz$

$u' = 1, \quad v = e^z$

$v_1 = e^z$

$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

$= (z+1)e^z - e^z + c$

$= ze^z + e^z - e^z + c = ze^z + c$

$\therefore f(z) = ze^z + c$

$u + iv = (x+iy)e^{x+iy} + c$

$= (x+iy)e^x \cdot e^{iy} + c = (x+iy)e^x(\cos y + i\sin y) + c$

$= e^x[x\cos y + ix\sin y + iy\cos y - y\sin y] + c$

$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) + c$

$\therefore v = e^x(x\sin y + y\cos y)$

(13) Given that $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, find the analytic function $f(z) = u + iv$. [N/D - 2014] [N/D - 2012]

Sol: Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\varphi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\varphi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - \sin 2z(2\sin 2z)}{(1-\cos 2z)^2} = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2} \quad (\because \sin^2 2z + \cos^2 2z = 1)$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1+\cos 2z)(1-\cos 2z)}{(1-\cos 2z)^2} \quad (\because a^2 - b^2 = (a+b)(a-b))$$

$$= \frac{(1-\cos 2z)[2\cos 2z - 2 - 2\cos 2z]}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z}$$

$$\left(\because \sin^2 z = \frac{1-\cos 2z}{2}\right)$$

$$= \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\varphi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)\cdot 0 - \sin 2x(\sinh 2y \cdot 2)}{(\cosh 2y - \cos 2x)^2} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\varphi_2(z,0) = \frac{-2\sin 2z \cdot 0}{(1-\cos 2z)^2} = 0$$

By Milne's Thomson method, $f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz$

$\therefore f(z) = \int -\operatorname{cosec}^2 z\,dz - i\int 0.\,dz = -\int \operatorname{cosec}^2 z\,dz = \cot z + c$

(14) Show that $u = e^{-x}(x\cos y + y\sin y)$ is harmonic function. Hence find the analytic function $f(z) = u + iv$.

Sol: Given $u = e^{-x}(x\cos y + y\sin y) = e^{-x}x\cos y + e^{-x}y\sin y$

$u_x = \frac{\partial u}{\partial x} = \cos y\left[e^{-x}.1 + x(-e^{-x})\right] + y\sin y(-e^{-x})$

$u_x = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y = \varphi_1(x,y)$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = -e^{-x}\cos y - \cos y(x(-e^{-x}) + e^{-x}) + e^{-x}y\sin y$

$u_{xx} = -e^{-x}\cos y + xe^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y = -2e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y$

$u_y = \frac{\partial u}{\partial y} = e^{-x}x(-\sin y) + e^{-x}(y\cos y + \sin y) = -e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y = \varphi_2(x,y)$

$u_{yy} = \frac{\partial^2 u}{\partial y^2} = -e^{-x}x\cos y + e^{-x}(y(-\sin y) + \cos y) + e^{-x}\cos y$

$u_{yy} = -e^{-x}x\cos y - e^{-x}y\sin y + e^{-x}\cos y + e^{-x}\cos y = -e^{-x}x\cos y - e^{-x}y\sin y + 2e^{-x}\cos y$

$\therefore u_{xx} + u_{yy} = -2e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}x\cos y - e^{-x}y\sin y + 2e^{-x}\cos y = 0$

Hence u is harmonic.

$\varphi_1(z,0) = e^{-z} - ze^{-z} = e^{-z}(1-z)$

$\varphi_2(z,0) = 0$

By Milne's Thomson method, $f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$

$u = 1-z,\ dv = e^{-z}dz$

$u' = -1 \quad v = \frac{e^{-z}}{-1}$

$v_1 = e^{-z}$

$\therefore f(z) = \int e^{-z}(1-z)\,dz = (1-z)\frac{e^{-z}}{-1} + e^{-z} + c$

$= e^{-z}(1-1+z) + c = ze^{-z} + c$

(15) Can $v = \tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function $f(z) = u + iv$, taking v as the imaginary part & hence find u. [A/M – 2016]

Sol: Given $v = \tan^{-1}\left(\frac{y}{x}\right)$

$v_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \varphi_2(x,y)\left(\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right)$

$v_{xx} = \frac{(x^2+y^2).0 + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$

$\varphi_2(z,0) = 0$

$\varphi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}$

$v_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \varphi_1(x,y)$

$$v_{yy} = \frac{(x^2+y^2).0 - x(+2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$\therefore v_{xx} + v_{yy} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$. Hence v is harmonic. Every harmonic function is the real part or the imaginary part of some analytic function. $\therefore$ Given v is the imaginary part of an analytic function.

By Milne - Thomson's method,

$$f(z) = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz$$

$$= \int \frac{1}{z}\,dz = \log z + c$$

$$u+iv = \log(x+iy) + c = \log(re^{i\theta}) + c = \log r + \log e^{i\theta} + c = \log\sqrt{x^2+y^2} + i\theta + c$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + c \qquad \therefore u = \frac{1}{2}\log(x^2+y^2)$$

(16) Find the analytic function $f(z) = u+iv$ if $u - v = e^x(\cos y - \sin y)$. [April /May - 2018] [M/J - 2014]

Sol: WKT $f(z) = u+iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2) $\Rightarrow f(z) + if(z) = u + iv + iu - v = (u-v) + i(u+v)$

$F(z) = f(z)(1+i) = (u-v) + i(u+v) = U + iV$

$U = u - v = e^x(\cos y - \sin y)$

$\varphi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$; $\varphi_1(z,0) = e^z(1-0) = e^z$

$\varphi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$; $\varphi_2(z,0) = e^z(-0-1) = -e^z$

By Milne's method, $F(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$

$\therefore F(z) = \int e^z\,dz - i\int -e^z\,dz = e^z + ie^z + c = e^z(1+i) + c$

$f(z)(1+i) = e^z(1+i) + c \Rightarrow f(z) = e^z + c_1$

(17) Find the analytic function $f(z) = u+iv$, given that $2u + 3v = e^x(\cos x - \sin y)$. [A/M - 2017]

Sol: WKT $f(z) = u + iv$ — (1)

$-if(z) = -iu - i^2 v = -iu + v$ — (2)

(1) $\times 2 \Rightarrow 2f(z) = 2u + i2v$

(2) $\times 3 \Rightarrow -3if(z) = 3v - i3u$

$2f(z) - i3f(z) = 2u + 3v + i(2v - 3u)$

$F(z) = (2-3i)f(z) = 2u + 3v + i(2v - 3u) = U + iV$

$U = 2u + 3v = e^x(\cos x - \sin y) = e^x\cos x - e^x \sin y$

$\varphi_1(x,y) = \frac{\partial U}{\partial x} = e^x(-\sin x) + \cos x e^x - e^x \sin y = -e^x\sin x + e^x\cos x - e^x\sin y$

$\varphi_1(z,0) = -e^z \sin z + e^z \cos z - e^z . 0 = -e^z \sin z + e^z \cos z$

$\varphi_2(x,y) = \frac{\partial u}{\partial y} = 0 - e^x \cos y = -e^x \cos y$

$\varphi_2(z,0) = -e^z . 1 = -e^z$

By Milne-Thomson method, $F(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz$

$\therefore F(z) = \int (-e^z \sin z + e^z \cos z)dz - i\int -e^z dz$

$= -\int e^z \sin z dz + \int e^z \cos z dz + i\int e^z dz$

$\left[\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \right.$

$\left. \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \right]$

$= -\frac{e^z}{2}(\sin z - \cos z) + \frac{e^z}{2}(\cos z + \sin z) + ie^z + c$

$(2-3i)f(z) = \frac{e^z}{2}(\cos z - \sin z) + \frac{e^z}{2}(\cos z + \sin z) + ie^z + c$

$\therefore f(z) = \frac{e^z}{2(2-3i)}(\cos z - \sin z) + \frac{e^z}{2(2-3i)}(\cos z + \sin z) + \frac{i}{2-3i}e^z + c_1$

$= \frac{2+3i}{2(4-9)}e^z(\cos z - \sin z) + \frac{2+3i}{2(4-9)}e^z(\cos z + \sin z) + \frac{i(2+3i)}{4-9}e^z + c_1$

$= \frac{2+3i}{10}e^z(\sin z - \cos z) - \frac{2+3i}{10}e^z(\cos z + \sin z) - \frac{(2i-3)}{5}e^z + c_1$

$= \frac{2+3i}{10}e^z(\sin z - \cos z - \cos z - \sin z) - \frac{(2i-3)}{5}e^z + c_1$

$\therefore f(z) = -\frac{(2+3i)}{5}e^z \cos z - \frac{(2i-3)}{5}e^z + c_1$

(18) Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v. [NID – 2015]

Sol: Given $u = e^{x^2-y^2}\cos 2xy$

$\varphi_1(x,y) = \frac{\partial u}{\partial x} = e^{x^2-y^2}(-\sin 2xy . 2y) + \cos 2xy . e^{x^2-y^2} . 2x$

$\varphi_1(z,0) = 2z\cos 0 . e^{z^2} = 2ze^{z^2}$

$\varphi_2(x,y) = \frac{\partial u}{\partial y} = e^{x^2-y^2}(-\sin 2xy . 2x) + \cos 2xy . e^{x^2-y^2} . (-2y)$

$\varphi_2(z,0) = -2ze^{z^2}\sin 0 = 0$

By Milne-Thomson method,

Put $z^2 = t$

$2z dz = dt$

$f(z) = \int \varphi_1(z,0)dz - i\int \varphi_2(z,0)dz$

$= \int 2ze^{z^2}dz = \int e^t dt = e^t + c = e^{z^2} + c$

$$u+iv = e^{(x+iy)^2}+c = e^{x^2-y^2+2ixy}+c = e^{x^2-y^2}\cdot e^{i2xy}+c$$

$$= e^{x^2-y^2}\left[\cos 2xy + i\sin 2xy\right]+c$$

$$= e^{x^2-y^2}\cos 2xy + i e^{x^2-y^2}\sin 2xy + c$$

$$\therefore v = e^{x^2-y^2}\sin 2xy.$$

(19) Find the analytic function $f(z) = P+iQ$, if $P-Q = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

[M/J-2009]

Sol: WKT $f(z) = P+iQ$ —— (1)

$if(z) = iP - Q$ —— (2)

$(1)+(2) \Rightarrow f(z)+if(z) = P+iQ+iP-Q = (P-Q)+i(P+Q)$

$$F(z) = f(z)(1+i) = (P-Q)+i(P+Q) = U+iV$$

$$U = P-Q = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\varphi_1(x,y) = \frac{\partial U}{\partial x} = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\varphi_1(z,0) = \frac{(1-\cos 2z)2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2} = \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2} = \frac{2(\cos 2z - 1)}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z}$$

$$\varphi_2(x,y) = \frac{\partial U}{\partial y} = \frac{(\cosh 2y - \cos 2x)\cdot 0 - \sin 2x \sinh 2y \cdot 2}{(\cosh 2y - \cos 2x)^2} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$\varphi_2(z,0) = 0 \quad (\because \sinh 2y = 0)$

By Milne-Thomson method, $F(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$

$$\therefore F(z) = \int \frac{-2}{1-\cos 2z}\,dz = \int \frac{-2}{2\sin^2 z}\,dz = -\int \operatorname{cosec}^2 z\,dz = -(-\cot z)+c$$

$$F(z) = \cot z + c \Rightarrow f(z)(1+i) = \cot z + c$$

$$\therefore f(z) = \frac{1}{1+i}\cot z + c, = \frac{1-i}{1-i^2}\cot z + c$$

$$\therefore f(z) = \frac{1-i}{2}\cot z + c$$

(20) If $w=f(z)$ is analytic prove that $\frac{dw}{dz}=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}$ where $z=x+iy$ & prove that $\frac{\partial^2 w}{\partial z\partial \bar{z}}=0$. [AIM - 2011]

Sol. Given $w=f(z)=u+iv$ & $f(z)$ is analytic, we have $u_x=v_y$, $u_y=-v_x$.

$$\frac{dw}{dz}=f'(z)=u_x+iv_x=v_y-iu_y=-i\left(u_y+\frac{1}{-i}v_y\right)=-i(u_y+iv_y)$$

$$\Rightarrow u_x+iv_x=-i(u_y+iv_y)=\frac{dw}{dz}$$

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=\frac{dw}{dz}$$

$$\frac{\partial}{\partial x}(u+iv)=-i\frac{\partial}{\partial y}(u+iv)=\frac{dw}{dz}$$

$$\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}=\frac{dw}{dz}$$

WKT $\frac{\partial w}{\partial \bar{z}}=0$ $\therefore \frac{\partial^2 w}{\partial z\partial \bar{z}}=0$. Also $\frac{\partial^2 w}{\partial \bar{z}\partial z}=0$

(21) If $u=x^2-y^2$, $v=\frac{y}{x^2+y^2}$, prove that u & v are harmonic functions but $f(z)=u+iv$ is not an analytic function. [N/D 2015]

Sol: Given $u=x^2-y^2$, $v=\frac{y}{x^2+y^2}$

$u_x=2x$, $u_{xx}=2$, $v_x=\frac{(x^2+y^2).0-y.2x}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}$

$u_y=-2y$, $u_{yy}=-2$

$$v_{xx}=\frac{(x^2+y^2)^2.-2y-(-2xy)2(x^2+y^2).2x}{(x^2+y^2)^4}$$

$$v_{xx}=\frac{-2y(x^2+y^2)^2+8x^2y(x^2+y^2)}{(x^2+y^2)^4}=\frac{-2y(x^2+y^2)+8x^2y}{(x^2+y^2)^3}=\frac{-2yx^2-2y^3+8x^2y}{(x^2+y^2)^3}$$

$$=\frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$$v_y=\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$v_{yy}=\frac{(x^2+y^2)^2.(-2y)-(x^2-y^2).2(x^2+y^2).2y}{(x^2+y^2)^4}=\frac{(x^2+y^2)(-2y)-4y(x^2-y^2)}{(x^2+y^2)^3}$$

$$=\frac{-2x^2y-2y^3-4x^2y+4y^3}{(x^2+y^2)^3}=\frac{-6x^2y+2y^3}{(x^2+y^2)^3}$$

$\therefore u_{xx}+u_{yy}=2-2=0$. Hence u is harmonic.

$$v_{xx}+v_{yy}=\frac{1}{(x^2+y^2)^3}\left[6x^2y-2y^3-6x^2y+2y^3\right]=0$$

Hence v is harmonic.

$u_x=2x\neq v_y$ & $u_y=-2y\neq -v_x$. Hence $f(z)$ is not analytic.

(22) Prove that $u=e^{-y}\cos x$ & $v=e^{-x}\sin y$ satisfy Laplace equations but that $u+iv$ is not an analytic function of z. [M/J-2011]

Sol: Given $u=e^{-y}\cos x$ & $v=e^{-x}\sin y$

$u_x=e^{-y}(-\sin x)=-e^{-y}\sin x$, $u_{xx}=-e^{-y}\cos x$

$u_y=-e^{-y}\cos x$, $u_{yy}=e^{-y}\cos x$

$\therefore u_{xx}+u_{yy}=-e^{-y}\cos x+e^{-y}\cos x=0$. Hence u satisfies Laplace eqn.

$v_x=-e^{-x}\sin y$, $v_{xx}=e^{-x}\sin y$

$v_y=e^{-x}\cos y$, $v_{yy}=-e^{-x}\sin y$

$\therefore v_{xx}+v_{yy}=e^{-x}\sin y-e^{-x}\sin y=0$. Hence v satisfies Laplace eqn.

$u_x=-e^{-y}\sin x\neq v_y$ & $u_y=-e^{-y}\cos x\neq -v_x$. Hence $u+iv$ is not an analytic func. of z.

(23) If $u(x,y)$ & $v(x,y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ is an analytic function of $z=x+iy$.

Sol. As u & v are harmonic, the following are true in R.

(i) $u_{xx}+u_{yy}=0$ (ii) $v_{xx}+v_{yy}=0$

(iii) Second order partial derivatives of u & v are continuous.

Let $U=u_y-v_x$ & $V=u_x+v_y$

Then $U_x=u_{xy}-v_{xx}$ $\quad V_x=u_{xx}+v_{xy}$

$U_y=u_{yy}-v_{yx}$ $\quad V_y=u_{yx}+v_{yy}$

$$U_x=u_{xy}-v_{xx}=u_{xy}-(-v_{yy}) \quad (\because \text{by (ii)})$$
$$=u_{xy}+v_{yy}=u_{yx}+v_{yy}=V_y$$

$\therefore U_x=V_y$

$$U_y=u_{yy}-v_{yx}=-u_{xx}-v_{yx} \quad (\because \text{by (i)})$$
$$=-(u_{xx}+v_{yx})=-(u_{xx}+v_{xy})=-V_x$$

$\therefore U_y=-V_x$

Further, U_x, U_y, V_x, V_y are continuous in R by (iii). Hence by sufficiency conditions of analyticity $U+iV$ is an analytic function of z.

(24) Prove that the real & imaginary parts of an analytic function are harmonic functions. [A/M-2014]

Sol: Let $f(z)=u+iv$ be an analytic function.

$\Rightarrow u_x = v_y$ —(1) & $u_y = -v_x$ —(2) by Cauchy-Riemann equations.

Differentiating (1) & (2) partially w.r.t. x, we get

$u_{xx} = v_{xy}$ —(3) & $u_{xy} = -v_{xx}$ —(4)

Differentiating (1) & (2) partially w.r.t. y, we get

$u_{yx} = v_{yy}$ —(5) & $u_{yy} = -v_{yx}$ —(6)

(3)+(6) $\Rightarrow u_{xx}+u_{yy} = v_{xy}-v_{yx} = 0$ $(\because v_{xy}=v_{yx})$

(5)−(4) $\Rightarrow u_{yx}-u_{xy} = v_{yy}-(-v_{xx}) \Rightarrow v_{xx}+v_{yy}=0$ $(\because u_{xy}=u_{yx})$

$\therefore u$ & v satisfy the Laplace equation.

(25) Find the analytic function $u+iv$, if $u=(x-y)(x^2+4xy+y^2)$. Also find the conjugate harmonic function v.

Sol: Given $u=(x-y)(x^2+4xy+y^2) = x^3+4x^2y+xy^2-x^2y-4xy^2-y^3$

$u = x^3+3x^2y-3xy^2-y^3$

$\varphi_1(x,y) = u_x = 3x^2+6xy-3y^2$ $\qquad$ $\varphi_2(x,y) = u_y = 3x^2-6xy-3y^2$

$\varphi_1(z,0) = 3z^2$ $\qquad$ $\varphi_2(z,0) = 3z^2$

By Milne-Thomson method, $f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz$

$\therefore f(z) = \int 3z^2 dz - i\int 3z^2 dz = 3\left(\frac{z^3}{3}\right) - i\,3\left(\frac{z^3}{3}\right) + c = z^3 - iz^3 + c$

$f(z) = z^3(1-i)+c \Rightarrow u+iv = (x+iy)^3(1-i)+c$

$u+iv = (x^3+3x^2(iy)+3x(iy)^2+(iy)^3)(1-i)+c$

$= (x^3+i3x^2y-3xy^2-iy^3)(1-i)+c$

$= x^3+i3x^2y-3xy^2-iy^3-ix^3+3x^2y+i3xy^2-y^3+c$

$= x^3-3xy^2+3x^2y-y^3+i(3x^2y-y^3-x^3+3xy^2)+c$

$\therefore v = 3x^2y-y^3-x^3+3xy^2$

(26) Show that the function $u=e^{-2xy}\sin(x^2-y^2)$ is a real part of an analytic function. Also find its conjugate harmonic function v & express $f(z)=u+iv$ as function of z. [N/D 2015] [N/D-2013]

Sol: Given $u=e^{-2xy}\sin(x^2-y^2)$

$\varphi_1(x,y) = u_x = e^{-2xy}\cos(x^2-y^2).2x + \sin(x^2-y^2)\,e^{-2xy}.(-2y)$

$$u_x = 2xe^{-2xy}\cos(x^2-y^2) - 2ye^{-2xy}\sin(x^2-y^2)$$

$$\varphi_1(z,0) = 2ze^0\cos z^2 = 2z\cos z^2$$

$$u_y = e^{-2xy}\cos(x^2-y^2).(-2y) + \sin(x^2-y^2)e^{-2xy}(-2x) = \varphi_2(x,y)$$

$$u_y = -2ye^{-2xy}\cos(x^2-y^2) - 2xe^{-2xy}\sin(x^2-y^2)$$

$$\varphi_2(z,0) = -2ze^0\sin z^2 = -2z\sin z^2$$

$$u_{xx} = 2x\left[e^{-2xy}.-\sin(x^2-y^2).2x + \cos(x^2-y^2).e^{-2xy}.(-2y)\right] + e^{-2xy}\cos(x^2-y^2).2$$
$$-2y\left[e^{-2xy}\cos(x^2-y^2).2x + \sin(x^2-y^2)e^{-2xy}.(-2y)\right] - 2e^{-2xy}\sin(x^2-y^2).0$$

$$u_{xx} = -4x^2e^{-2xy}\sin(x^2-y^2) - 4xye^{-2xy}\cos(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2)$$
$$-4xye^{-2xy}\cos(x^2-y^2) + 4y^2e^{-2xy}\sin(x^2-y^2)$$
$$= -4x^2e^{-2xy}\sin(x^2-y^2) - 8xye^{-2xy}\cos(x^2-y^2) + 2e^{-2xy}\cos(x^2-y^2)$$
$$+4y^2e^{-2xy}\sin(x^2-y^2)$$

$$\therefore u_{xx} = 2e^{-2xy}\sin(x^2-y^2)\left[-2x^2+2y^2\right] + 2e^{-2xy}\cos(x^2-y^2)\left[-4xy+1\right] \quad \text{— (1)}$$

$$u_{yy} = -2y\left[e^{-2xy}.-\sin(x^2-y^2)(-2y) + e^{-2xy}\cos(x^2-y^2)(-2x)\right] + e^{-2xy}\cos(x^2-y^2)(-2)$$
$$-2x\left[e^{-2xy}\cos(x^2-y^2)(-2y) + \sin(x^2-y^2)e^{-2xy}(-2x)\right] + e^{-2xy}\sin(x^2-y^2).(-2).0$$
$$= -4y^2e^{-2xy}\sin(x^2-y^2) + 4xye^{-2xy}\cos(x^2-y^2) - 2e^{-2xy}\cos(x^2-y^2)$$
$$+4xye^{-2xy}\cos(x^2-y^2) + 4x^2e^{-2xy}\sin(x^2-y^2)$$
$$= 2e^{-2xy}\sin(x^2-y^2)\left[-2y^2+2x^2\right] + 2e^{-2xy}\cos(x^2-y^2)\left[4xy-1\right] \quad \text{— (2)}$$

From (1) & (2), $u_{xx}+u_{yy}=0 \Rightarrow u$ is harmonic. Every harmonic function is the real part or the imaginary part of some analytic function. $\therefore$ Given is a real part of an analytic function.

By Milne-Thomson method,

Put $z^2=t$
$2z\,dz = dt$

$$f(z) = \int\varphi_1(z,0)dz - i\int\varphi_2(z,0)dz$$
$$= \int 2z\cos z^2 dz + i\int 2z\sin z^2 dz$$

$$\therefore f(z) = \int\cos t\,dt + i\int\sin t\,dt = \sin t - i\cos t + c = \sin z^2 - i\cos z^2 + c = -i\left(\cos z^2 + \frac{1}{-i}\sin z^2\right) + c$$
$$= -i(\cos z^2 + i\sin z^2) + c$$

$$u+iv = -ie^{+iz^2} + c = -ie^{+i(x+iy)^2} + c = -ie^{+i(x^2-y^2+i2xy)} + c$$
$$= (-i)e^{+i(x^2-y^2)-2xy} + c = -ie^{+i(x^2-y^2)}e^{-2xy} + c$$

$u+iv = e^{-2xy}\left(\cos(x^2-y^2)+i\sin(x^2-y^2)\right)(-i)+c$

$= e^{-2xy}\left(-i\cos(x^2-y^2)+\sin(x^2-y^2)\right)+c$

$= e^{-2xy}\sin(x^2-y^2) - ie^{-2xy}\cos(x^2-y^2)+c$

$\therefore v = -e^{-2xy}\cos(x^2-y^2)$

(27) Prove that $u = x^2-y^2$ & $v = \dfrac{-y}{x^2+y^2}$ are harmonic functions but not harmonic conjugates. ((or) but $u+iv$ is not regular). [N/D - 2014]

Sol: Given $u = x^2-y^2$ & $v = \dfrac{-y}{x^2+y^2}$

$u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$, $u_{yy} = -2$

$u_{xx}+u_{yy} = 2-2 = 0$. Hence u is harmonic.

$v_x = \dfrac{(x^2+y^2).0 + y(2x)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}$, $v_y = \dfrac{(x^2+y^2).(-1)+y(2y)}{(x^2+y^2)^2} = \dfrac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$

$v_y = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$$v_{xx} = \frac{(x^2+y^2)^2.2y - 2xy.2(x^2+y^2).2x}{(x^2+y^2)^4} = \frac{(x^2+y^2)2y - 8x^2y}{(x^2+y^2)^3} = \frac{2x^2y+2y^3-8x^2y}{(x^2+y^2)^3}$$

$$v_{xx} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$v_{yy} = \frac{(x^2+y^2)^2.2y - (-x^2+y^2)2(x^2+y^2).2y}{(x^2+y^2)^4} = \frac{(x^2+y^2)2y - 4y(-x^2+y^2)}{(x^2+y^2)^3}$$

$$= \frac{2x^2y+2y^3+4x^2y-4y^3}{(x^2+y^2)^3} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$\therefore v_{xx}+v_{yy} = \dfrac{2y^3-6x^2y}{(x^2+y^2)^3} + \dfrac{6x^2y-2y^3}{(x^2+y^2)^3} = 0$. Hence v is harmonic.

$u_x = 2x \neq v_y$ & $u_y = -2y \neq v_x$. Hence $u+iv$ is not an analytic (regular) function. $\therefore$ u & v are not harmonic conjugates.

Bilinear transformation: (Linear fractional transformation (or) Mobius transformation)

The transformation $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

Result: The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Cross ratio: Given 4 points z_1, z_2, z_3, z_4 in this order, the ratio $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

---

(28) Find the bilinear transformation which maps the pts $z=-1,0,1$ onto the points $w=-1,-i,1$. Show that under this transformation the upper half of the z-plane maps onto the interior of the unit circle $|w|=1$. [April / may - 2018] [A/M - 2017]

Sol: Given $z_1=-1$, $z_2=0$, $z_3=1$, $w_1=-1$, $w_2=-i$, $w_3=1$

Bilinear transformation: $\dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$

$$\frac{(z+1)(0-1)}{(z-1)(0+1)} = \frac{(w+1)(-i-1)}{(w-1)(-i+1)} \Rightarrow \frac{-(z+1)}{z-1} = \frac{-(w+1)(i+1)}{(w-1)(-i+1)}$$

$$(z+1)(w-1)(-i+1) = (w+1)(i+1)(z-1)$$

$$(wz-z+w-1)(-i+1) = (w+1)(iz-i+z-1)$$

$$-iwz+\cancel{iz}-\cancel{iw}+i+\cancel{wz}-z+w-1 = iwz-\cancel{iw}+\cancel{wz}-w+\cancel{iz}-i+z-1$$

$$-iwz-iwz+i+i-z-z+w+w=0$$

$$-2iwz+2i-2z+2w=0 \Rightarrow -iwz+i-z+w=0 \quad \text{—(1)}$$

$$-iwz+w = z-i$$

$$w(-iz+1) = z-i$$

$$w = \frac{z-i}{1-iz}$$

From (1), $-iwz-z=-w-i \Rightarrow -z(iw+1) = -(w+i)$

$$z = \frac{w+i}{iw+1}$$

$$x+iy = \frac{u+iv+i}{i(u+iv)+1} = \frac{u+iv+i}{iu-v+1} = \frac{u+iv+i}{1-v+iu} = \frac{(u+iv+i)(1-v-iu)}{(1-v+iu)(1-v-iu)}$$

$$= \frac{(u+i(v+1))(1-v-iu)}{(1-v)^2-(iu)^2} = \frac{u(1-v)-iu^2+i(v+1)(1-v)+u(v+1)}{(1-v)^2+u^2}$$

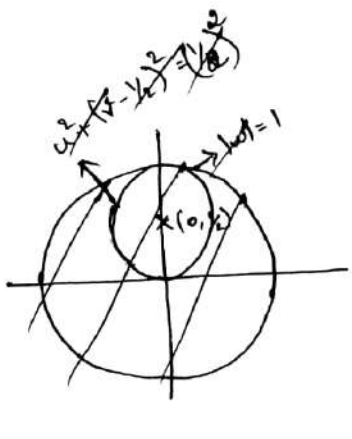
$$x+iy = \frac{u-uv-iu^2-i(v+1)(v-1)+uv+u}{1+v^2-2v+u^2} = \frac{2u-iu^2-i(v^2-1)}{u^2+v^2-2v+1}$$

$$x+iy = \frac{2u-i(u^2+v^2-1)}{u^2+v^2-2v+1} = \frac{2u}{u^2+v^2-2v+1} - i\,\frac{(u^2+v^2-1)}{u^2+v^2-2v+1}$$

From this $y = -\dfrac{(u^2+v^2-1)}{u^2+v^2-2v+1} = \dfrac{1-u^2-v^2}{u^2+v^2-2v+1}$

Given $y>0$ (Upper half of the z-plane)

$$\Rightarrow \frac{1-u^2-v^2}{u^2+v^2-2v+1} > 0 \Rightarrow 1-u^2-v^2 > 0$$



$u^2+v^2<1$

$\Rightarrow |w|<1$

$\cancel{-2u^2-2v^2+2v>0}$

$\cancel{-u^2-v^2+v>0}$

$\cancel{u^2+v^2-v<0}$

$\cancel{u^2+v^2-2\frac{1}{2}v+(\frac{1}{2})^2-(\frac{1}{2})^2<0} \;\; \cancel{\Rightarrow u^2+(v-\frac{1}{2})^2-\frac{1}{4}<0}$

$\cancel{u^2+(v-\frac{1}{2})^2<(\frac{1}{2})^2}$ ~~which is an interior of the circle whose centre is $(0,\frac{1}{2})$ & radius is $\frac{1}{2}$~~

Hence the upper half of the z-plane maps onto the interior of the unit circle $|w|=1$.

(29) Find the bilinear transformation which maps the points $z=1, i, -1$ onto the points $w=i, 0, -i$. Hence find the image of $|z|<1$. [N/D – 2014] [M/J – 2011]

Sol: Given $z_1=1$, $z_2=i$, $z_3=-1$, $w_1=i$, $w_2=0$, $w_3=-i$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(w-i)(0+i)}{(w+i)(0-i)}$$

$$\frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{w-i}{-(w+i)}$$

$$-(z-1)(i+1)(w+i) = (w-i)(z+1)(i-1)$$

$$-(iz+z-i-1)(w+i) = (wz+w-iz-i)(i-1)$$

$$-(iwz+wz-iw-w-z+iz+1-i) = iwz+iw+z+1-wz-w+iz+i$$

$$-iwz-wz+i\cancel{w}+w+\cancel{z}-iz-1+\cancel{i} = iwz+i\cancel{w}+\cancel{z}+1-wz-w+iz+\cancel{i}$$

$-2iwz+2w-2iz-2=0$

$-iwz+w-iz-1=0$ —— (1)

$w(1-iz)=iz+1 \Rightarrow w=\dfrac{iz+1}{1-iz}$

From (1), $-iwz-iz=1-w \Rightarrow z(-iw-i)=1-w$

$$z=\frac{1-w}{-iw-i}=\frac{1-(u+iv)}{-i(u+iv)-i}$$

$$z=\frac{1-u-iv}{-iu+v-i}$$

Given: $|z|<1 \Rightarrow \left|\dfrac{1-u-iv}{-iu+v-i}\right|<1$

$$|1-u-iv|<|v-i(1+u)|$$

$$\sqrt{(1-u)^2+(-v)^2}<\sqrt{v^2+(-(1+u))^2}$$

$$(1-u)^2+v^2<v^2+(1+u)^2$$

$$1+u^2-2u+v^2<v^2+1+u^2+2u$$

$$\cancel{1}+\cancel{u^2}-2u+\cancel{v^2}-\cancel{v^2}-\cancel{1}-\cancel{u^2}-2u<0$$

$$-4u<0 \Rightarrow 4u>0 \Rightarrow u>0$$

(30) Find the bilinear transformation which maps the points $z=1, i, -1$ onto the points $w=0, 1, \infty$. Find also the pre-image of $|w|=1$ under this bilinear transformation. [A/M – 2014] [M/J - 2014]

Sol: Given $z_1=1, z_2=i, z_3=-1, w_1=0, w_2=1, w_3=\infty$.

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}=\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}=\frac{w-w_1}{w_2-w_1}$$

$$\frac{(z-1)(i+1)}{(z+1)(i-1)}=\frac{w-0}{1-0}=w$$

$$(z-1)(i+1)=w(z+1)(i-1)$$

$$iz+z-i-1=w(iz-z+i-1)$$

$$\therefore w=\frac{iz+z-i-1}{iz-z+i-1}$$

$$w = \frac{(i+1)z - i - 1}{(i-1)z + i - 1}$$

Given: $|w| = 1$

$$\left| \frac{(i+1)z - i - 1}{(i-1)z + i - 1} \right| = 1$$

$$|(i+1)z - i - 1| = |(i-1)z + i - 1|$$

$$|(i+1)(x+iy) - i - 1| = |(i-1)(x+iy) + i - 1|$$

$$|ix - y + x + iy - i - 1| = |ix - y - x - iy + i - 1|$$

$$|x - y - 1 + i(x+y-1)| = |-x - y - 1 + i(x - y + 1)|$$

$$\sqrt{(x-y-1)^2 + (x+y-1)^2} = \sqrt{(-x-y-1)^2 + (x-y+1)^2}$$

$$(x-y-1)^2 + (x+y-1)^2 = (x+y+1)^2 + (x-y+1)^2$$

$$\cancel{x^2} + \cancel{y^2} + \cancel{1} - \cancel{2xy} + \cancel{2y} - 2x + \cancel{x^2} + \cancel{y^2} + \cancel{1} + \cancel{2xy} - \cancel{2y} - 2x = \cancel{x^2} + \cancel{y^2} + \cancel{1} + \cancel{2xy} + \cancel{2y} + 2x + \cancel{x^2} + \cancel{y^2} + \cancel{1} - \cancel{2xy} - \cancel{2y} + 2x$$

$$-4x = 4x \Rightarrow 4x + 4x = 0 \Rightarrow 8x = 0 \Rightarrow x = 0$$

(31) Find the bilinear transformation which maps the points $z = 0, 1, \infty$ onto the points $w = i, 1, -i$. ['M/J-2010]

Sol: Given $z_1 = 0,\ z_2 = 1,\ z_3 = \infty,\ w_1 = i,\ w_2 = 1,\ w_3 = -i$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{z-z_1}{z_2-z_1} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{z-0}{1-0} = \frac{(w-i)(1+i)}{(w+i)(1-i)} = z$$

$$(w-i)(1+i) = z(w+i)(1-i)$$

$$w + iw - i + 1 = z(w - iw + i + 1) = zw - iwz + zi + z$$

$$w + iw - i + 1 - wz + iwz - iz - z = 0$$

$$w + iw - wz + iwz = i - 1 + iz + z$$

$$w(1 + i - z + iz) = i - 1 + iz + z \Rightarrow w = \frac{(i+1)z + i - 1}{z(i-1) + 1 + i}$$

(32) Find the bilinear transformation which maps the points $z = 0, 1, -1$ onto the points $w = -1, 0, \infty$. Find also the invariant points (fixed points) of the transformation. [N/D 2016]

<u>Sol:</u> Given $z_1 = 0$, $z_2 = 1$, $z_3 = -1$, $w_1 = -1$, $w_2 = 0$, $w_3 = \infty$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{w-w_1}{w_2-w_1}$$

$$\frac{(z-0)(1+1)}{(z+1)(1-0)} = \frac{w+1}{0+1} = w+1$$

$$\frac{2z}{(z+1)} = w+1 \Rightarrow 2z = (w+1)(z+1) \Rightarrow w+1 = \frac{2z}{z+1}$$

$$\therefore w = \frac{2z}{z+1} - 1 = \frac{2z-(z+1)}{z+1} = \frac{2z-z-1}{z+1} = \frac{z-1}{z+1}$$

<u>Fixed points:</u> $z = \frac{z-1}{z+1} \Rightarrow z(z+1) = z-1$

$$z^2+z-z+1 = 0$$

$$z^2+1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \sqrt{-1} = \pm i$$

$$\therefore z = \pm i$$

(33) Find the bilinear transformation which maps the points $z = 0, -1, i$ onto the points $w = i, 0, \infty$. Also find the image of the unit circle of the z-plane. [N/D-2013]

<u>Sol:</u> Given $z_1 = 0$, $z_2 = -1$, $z_3 = i$, $w_1 = i$, $w_2 = 0$, $w_3 = \infty$.

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{w-w_1}{w_2-w_1}$$

$$\frac{(z-0)(-1-i)}{(z-i)(-1-0)} = \frac{w-i}{0-i} \Rightarrow \frac{z(-1-i)}{-(z-i)} = \frac{w-i}{-i}$$

$$iz(-1-i) = (w-i)(z-i)$$

$$-iz+1z = wz-iw-iz-1 \Rightarrow wz-iw = -iz+z+iz+1$$

$$w(z-i) = z+1$$

$$\therefore w = \frac{z+1}{z-i} \quad \text{—} (1)$$

<u>To find:</u> Image of the unit circle of the z-plane.

<u>Given:</u> $|z| = 1$

From (1), $w(z-i) = z+1 \Rightarrow wz - iw - z = 1$

$$wz - z = 1 + iw$$

$$z(w-1) = 1+iw$$

$$z = \frac{1+iw}{w-1}$$

$$|z| = 1 \Rightarrow \left|\frac{1+iw}{w-1}\right| = 1 \Rightarrow |1+iw| = |w-1|$$

$$\Rightarrow |1+i(u+iv)| = |u+iv-1|$$

$$\Rightarrow |1+iu-v| = |u-1+iv|$$

$$\Rightarrow |1-v+iu| = |u-1+iv|$$

$$\sqrt{(1-v)^2+u^2} = \sqrt{(u-1)^2+v^2}$$

$$(1-v)^2+u^2 = (u-1)^2+v^2$$

$$1+v^2-2v+u^2 = u^2+1-2u+v^2$$

$$-2v+2u = 0$$

$$2(u-v) = 0 \Rightarrow u-v = 0 \Rightarrow u = v$$

---

(34) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = 0, 1, \infty$. Also show that the transformation maps interior of the unit circle of the z-plane onto the upper half of the w-plane.

<u>Sol:</u> Given $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{w-w_1}{w_2-w_1} \Rightarrow \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{w-0}{1-0}$$

$$(z-i)(i+1)=\omega(z+1)(i-1)$$
$$iz+z-i-1=\omega(iz-z+i-1) \quad \text{—} ①$$
$$\therefore \omega=\frac{iz+z-i-1}{iz-z+i-1}=\frac{(i+1)z-i-1}{(i-1)z+i-1}$$

From ①, $iz+z-i-1=i\omega z-\omega z+i\omega-\omega$
$$iz+z-i\omega z+\omega z=i\omega-\omega+i+1$$
$$z(i+1-i\omega+\omega)=i\omega-\omega+i+1$$
$$z=\frac{i\omega-\omega+i+1}{i+1-i\omega+\omega}$$

Given: $|z|<1$
$$\left|\frac{i\omega-\omega+i+1}{i+1-i\omega+\omega}\right|<1 \Rightarrow |i\omega-\omega+i+1|<|i+1-i\omega+\omega|$$
$$|i(u+iv)-(u+iv)+i+1|<|i+1-i(u+iv)+u+iv|$$
$$|iu-v-u-iv+i+1|<|i+1-iu+v+u+iv|$$
$$|1-u-v+i(u-v+1)|<|1+u+v+i(1-u+v)|$$
$$\sqrt{(1-u-v)^2+(u-v+1)^2}<\sqrt{(1+u+v)^2+(1-u+v)^2}$$
$$1+u^2+v^2-\cancel{2u}+2uv-2v+u^2+v^2+1-2uv-2v+\cancel{2u}<1+u^2+v^2+2u+2uv+2v$$
$$+1+u^2+v^2-2u-2uv+2v$$

$$2u^2+2v^2-4v+2<2+2u^2+2v^2+4v$$
$$u^2+v^2-2v+1<1+u^2+v^2+2v \Rightarrow 4v>0 \Rightarrow v>0 \text{ (upper half of the } \omega\text{-plane)}$$

Hence the interior of the unit circle of the z-plane maps onto the upper half of the ω-plane.

Conformal mapping:

A transformation that preserves angles between every pair of curves through a point, both in magnitude & sense, is said to be conformal at that point.

Isogonal: A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be isogonal at that point.

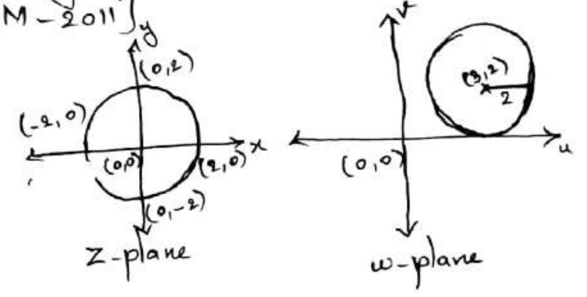
Note: ① A mapping $w=f(z)$ is said to be conformal at $z=z_0$, if $f'(z_0)\neq 0$.

② The point, at which the mapping $w=f(z)$ is not conformal, (ie) $f'(z)=0$ is called a critical point of the mapping.

(35) Find the image of $|z|=2$ under the mapping (i) $w=z+3+2i$ (ii) $w=3z$. [A/M-2011]

Sol: (i) Given $w=z+3+2i$

$$u+iv=x+iy+3+2i=x+3+i(y+2)$$



$\therefore u=x+3,\quad v=y+2$

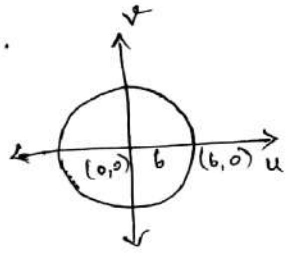
$x=u-3,\quad y=v-2$

Given: $|z|=2$ (ie) $x^2+y^2=2^2$

$$|x+iy|=2 \Rightarrow \sqrt{x^2+y^2}=2 \Rightarrow x^2+y^2=2^2 \Rightarrow x^2+y^2=4$$

$$\therefore (u-3)^2+(v-2)^2=4 \Rightarrow (u-3)^2+(v-2)^2=2^2$$

Hence the circle $x^2+y^2=4$ is mapped into $(u-3)^2+(v-2)^2=4$ in w-plane which is also a circle with centre $(3,2)$ & radius 2.



(ii) Given $w=3z$

$$u+iv=3(x+iy)=3x+i3y$$

$$\therefore u=3x,\quad v=3y \Rightarrow x=u/3,\quad y=v/3$$

Given: $|z|=2 \Rightarrow x^2+y^2=4$

$$\therefore \left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=4 \Rightarrow \frac{u^2}{9}+\frac{v^2}{9}=4 \Rightarrow u^2+v^2=36 \Rightarrow u^2+v^2=6^2$$

Hence the circle $x^2+y^2=4$ is mapped into $u^2+v^2=36$ in w-plane which is also a circle with centre $(0,0)$ & radius 6.

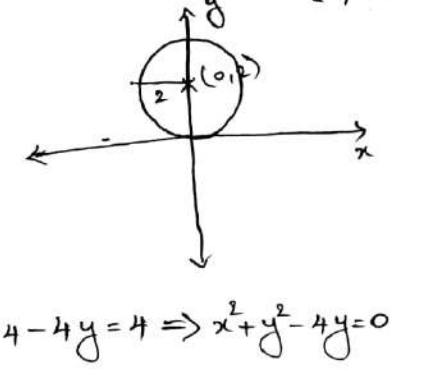
(36) Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$. [April/May-2018]

Sol: Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$x+iy=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

$$\therefore x=\frac{u}{u^2+v^2} \quad \& \quad y=\frac{-v}{u^2+v^2}$$

z-plane



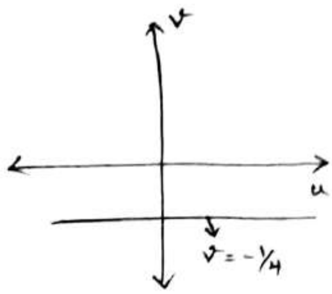
Given: $|z-2i|=2$

$$|x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$$

$$\sqrt{x^2+(y-2)^2}=2$$

$$x^2+(y-2)^2=4 \Rightarrow x^2+y^2+4-4y=4 \Rightarrow x^2+y^2-4y=0$$

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$



$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}=0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$u^2+v^2+4v(u^2+v^2)=0 \Rightarrow (u^2+v^2)(1+4v)=0$$

$$\Rightarrow 1+4v=0 \quad (\because u^2+v^2\neq 0)$$ which is a straight line in w-plane.

$$\Rightarrow v=-1/4$$

(37) Show that the transformation $w=\frac{1}{z}$ transforms in general, circles & straight lines into circles or straight lines. [N/D 2016] [N/D_2015] [A/M_2017] [N/D-2011]

Sol: Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$x+iy=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

$$\therefore x=\frac{u}{u^2+v^2}, \quad y=\frac{-v}{u^2+v^2}$$

The general equation is $a(x^2+y^2)+2gx+2fy+c=0$

$$a\left[\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right]+2g\left(\frac{u}{u^2+v^2}\right)+2f\left(\frac{-v}{u^2+v^2}\right)+c=0$$

$$a\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right)+2g\frac{u}{u^2+v^2}-2f\frac{v}{u^2+v^2}+c=0$$

$$a\frac{1}{u^2+v^2}+2g\frac{u}{u^2+v^2}-2f\frac{v}{u^2+v^2}+c=0$$

$$a+2gu-2fv+c(u^2+v^2)=0$$

$$c(u^2+v^2)+2gu-2fv+a=0$$

The transformed equation is $c(u^2+v^2)+2gu-2fv+a=0$

(i) $a\neq 0,\ c\neq 0 \Rightarrow$ circles not passing through the origin in z-plane map into circles not passing through the origin in the w-plane.

(ii) $a\neq 0,\ c=0 \Rightarrow$ circles through the origin in z-plane map onto straight lines not through the origin in the w-plane.

(iii) $a=0,\ c\neq 0 \Rightarrow$ the straight lines not through the origin in z-plane map onto circles through the origin in the w-plane.

(iv) $a=0,\ c=0 \Rightarrow$ straight lines through the origin in z-plane map onto straight lines through the origin in the w-plane.

(38) Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane to the upper half of the w-plane & also find the image of the unit circle of the z-plane. [N/D-2013] [N/D-2012] [M/J-2010]

Sol: Given $w = \frac{z}{1-z}$

$$w(1-z) = z \Rightarrow w - wz = z \Rightarrow wz + z = w \Rightarrow z(w+1) = w$$

$$\therefore z = \frac{w}{w+1} = \frac{u+iv}{u+iv+1} = \frac{u+iv}{u+1+iv} = \frac{(u+iv)(u+1-iv)}{(u+1+iv)(u+1-iv)}$$

$$= \frac{u(u+1) - iuv + iv(u+1) + v^2}{(u+1)^2 + v^2} = \frac{u(u+1) + v^2 + i(v(u+1) - uv)}{(u+1)^2 + v^2}$$

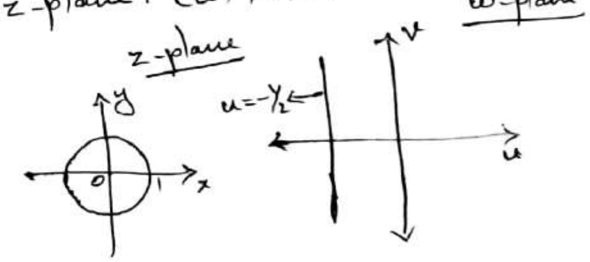
$$\therefore x + iy = \frac{u(u+1) + v^2 + i(uv + v - uv)}{(u+1)^2 + v^2} = \frac{u(u+1) + v^2 + iv}{(u+1)^2 + v^2}$$

$$\Rightarrow y = \frac{v}{(u+1)^2 + v^2}$$

Given: Upper half of the z-plane (ie) $y > 0$

$\frac{v}{(u+1)^2 + v^2} > 0 \Rightarrow v > 0$ (ie) upper half of the w-plane.

To find: Image of the unit circle of the z-plane. (ie) $|z| = 1$



$$\left|\frac{w}{w+1}\right| = 1 \Rightarrow |w| = |w+1|$$

$$|u+iv| = |u+iv+1|$$

$$\sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$u^2 + v^2 = (u+1)^2 + v^2 = u^2 + 1 + 2u + v^2$$

$2u + 1 = 0 \Rightarrow u = -\frac{1}{2}$ which is a straight line in w-plane.

(39) Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscates $r^2 = \cos 2\theta$. [N/D-2012] [M/J-2010]

Sol: Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$x + iy = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore x = \frac{1}{r}\cos\theta, \quad y = -\frac{1}{r}\sin\theta$$

Given: $x^2 - y^2 = 1$

$$\left(\frac{1}{r}\cos\theta\right)^2 - \left(\frac{-1}{r}\sin\theta\right)^2 = 1$$

$$\frac{\cos^2\theta}{r^2} - \frac{\sin^2\theta}{r^2} = 1 \Rightarrow \cos^2\theta - \sin^2\theta = r^2$$

$$\Rightarrow \frac{1+\cos 2\theta}{2} - \frac{1-\cos 2\theta}{2} = r^2$$

$$\Rightarrow 1+\cos 2\theta - 1 + \cos 2\theta = 2r^2$$

$\Rightarrow 2\cos 2\theta = 2r^2 \Rightarrow r^2 = \cos 2\theta$ which is lemniscate.

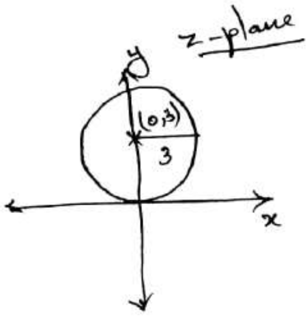
(40) Find the image of the circle $|z-3i|=3$ & the region $1<x<2$ under the map $w=\frac{1}{z}$. [M/J-2014]

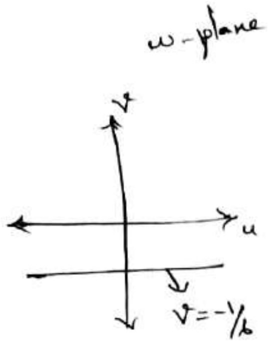
Sol: Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$x+iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Given: $|z-3i|=3$





$$|x+iy-3i| = 3 \Rightarrow |x+i(y-3)| = 3$$

$$\sqrt{x^2+(y-3)^2} = 3 \Rightarrow x^2+(y-3)^2 = 9$$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2} - 3\right)^2 = 9$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 9 + 6\frac{v}{u^2+v^2} = 9$$

$$u^2+v^2+6v(u^2+v^2) = 0 \times (u^2+v^2)^2 = 0$$

$$(u^2+v^2)(1+6v) = 0$$

$$\therefore 1+6v = 0 \quad (\because u^2+v^2 \neq 0)$$

$\Rightarrow v = -1/6$ which is a straight line in the w-plane.

Given: $1<x<2$

When $x=1$, $\frac{u}{u^2+v^2} = 1 \Rightarrow u = u^2+v^2$

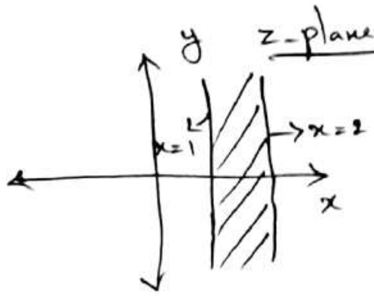
$$u^2+v^2-u=0 \Rightarrow u^2-u+v^2=0 \Rightarrow u^2 - 2\cdot\frac{1}{2}u + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + v^2 = 0$$

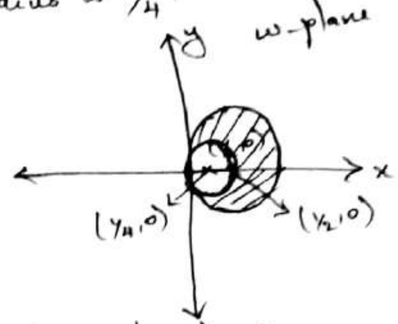
$(u-\frac{1}{2})^2 + v^2 = \frac{1}{4} \Rightarrow (u-\frac{1}{2})^2 + v^2 = (\frac{1}{2})^2$ — (1) which is a circle whose centre is $(\frac{1}{2}, 0)$ & radius is $\frac{1}{2}$.

When $x=2$, $\frac{u}{u^2+v^2}=2 \Rightarrow 2u^2+2v^2-u=0 \Rightarrow u^2+v^2-\frac{u}{2}=0$

$$u^2-2\cdot\frac{1}{2}\frac{u}{2}+v^2=0 \Rightarrow u^2-2u\left(\tfrac{1}{4}\right)+\left(\tfrac{1}{4}\right)^2-\left(\tfrac{1}{4}\right)^2+v^2=0$$

$\left(u-\tfrac{1}{4}\right)^2+v^2=\left(\tfrac{1}{4}\right)^2$ — (2) which is a circle whose centre is $(1/4, 0)$ & radius is $1/4$.





Hence the infinite strip $1<x<2$ is transformed into the region in between the circles (1) & (2) in the w-plane.

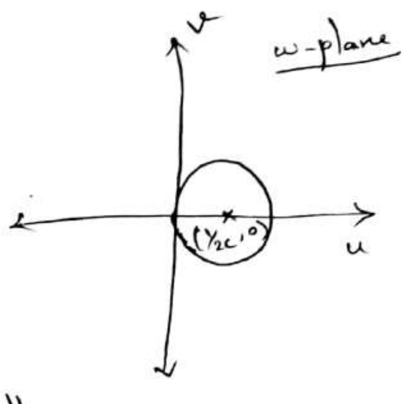
---

(41) Find the image of the half plane $x>c$, $c>0$ under $w=\frac{1}{z}$. Sketch graphically. Also find the fixed point of w. [M/J-2009]

Sol: Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$x+iy=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

$$\therefore x=\frac{u}{u^2+v^2},\quad y=-\frac{v}{u^2+v^2}$$



Given: $x>c$, $c>0$

$$\frac{u}{u^2+v^2}>c \Rightarrow u>c(u^2+v^2) \Rightarrow \frac{u}{c}>u^2+v^2$$

$$u^2+v^2-\frac{u}{c}<0 \Rightarrow u^2-2\cdot\frac{1}{2}\frac{u}{c}+v^2<0$$

$$u^2-2u\left(\frac{1}{2c}\right)+\left(\tfrac{1}{2c}\right)^2-\left(\tfrac{1}{2c}\right)^2+v^2<0$$

$\left(u-\frac{1}{2c}\right)^2+v^2<\left(\tfrac{1}{2c}\right)^2$ which is an interior of the circle whose centre is $(1/2c, 0)$ & radius is $\frac{1}{2c}$.

Fixed point: $z=\frac{1}{z} \Rightarrow z^2=1 \Rightarrow z=\sqrt{1}=\pm 1$

$$\therefore z=\pm 1$$

---

(42) Find the image of the lines $u=a$ & $v=b$ in w-plane into z-plane under the Transformation $z=\sqrt{w}$. [N/D 2015]

Scanned by CamScanner

Sol: Given $z = \sqrt{w}$

$$x+iy = \sqrt{u+iv} \Rightarrow (x+iy)^2 = u+iv \Rightarrow x^2 + (iy)^2 + i2xy = u+iv$$

$$\Rightarrow x^2 - y^2 + i2xy = u+iv$$

$$\therefore u = x^2 - y^2, \quad v = 2xy$$

Given: $u = a$

$x^2 - y^2 = a$ which is a hyperbola.

& $v = b \Rightarrow 2xy = b$ which is a rectangular hyperbola.

(43) Find the image of the region bounded by the lines $x=0$, $y=0$ & $x+y=1$ under the mappings $w = e^{i\pi/4} z$ & $w = z + (2+3i)$. [M/J - 2014]

Sol: Given: $w = e^{i\pi/4} z = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) z = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) z$

$$u+iv = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy) = \frac{x}{\sqrt{2}} + i\frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} - \frac{y}{\sqrt{2}}$$

$$u+iv = \frac{1}{\sqrt{2}}(x-y) + \frac{i}{\sqrt{2}}(x+y)$$

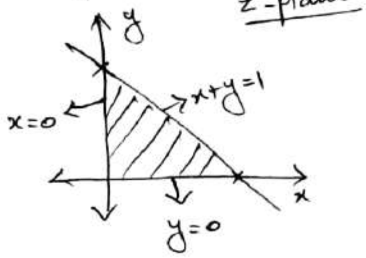
$$\therefore u = \frac{1}{\sqrt{2}}(x-y), \quad v = \frac{1}{\sqrt{2}}(x+y)$$

$x+y=1 \Rightarrow y = 1-x$

x:	0	1
y:	1	0

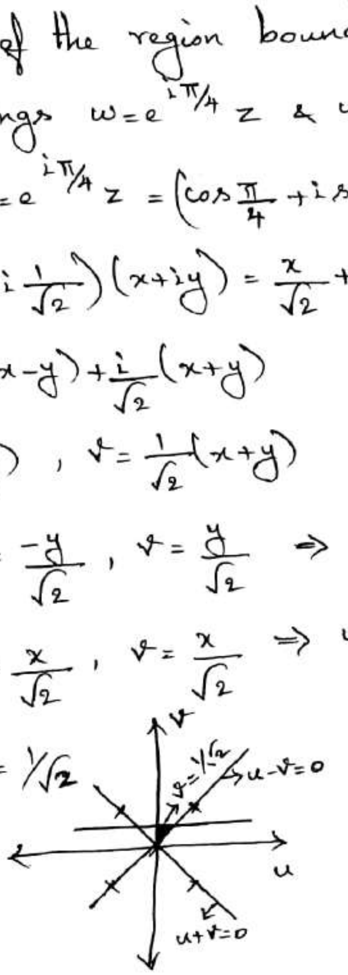
When $x=0$, $u = \frac{-y}{\sqrt{2}}$, $v = \frac{y}{\sqrt{2}} \Rightarrow u = -v \Rightarrow u+v=0$

z-plane



When $y=0$, $u = \frac{x}{\sqrt{2}}$, $v = \frac{x}{\sqrt{2}} \Rightarrow u = v \Rightarrow u - v = 0$

When $x+y=1$, $v = 1/\sqrt{2}$



$u+v=0 \Rightarrow v = -u$

u:	-1	0	1
v:	1	0	-1

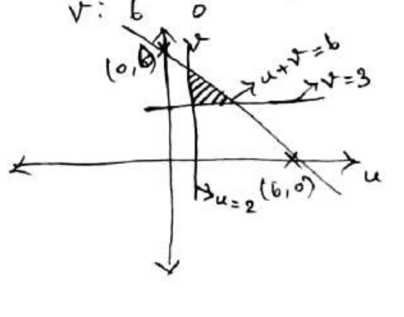
$u-v=0 \Rightarrow u=v$

u:	-1	0	1
v:	-1	0	1

$u+v=6 \Rightarrow v = 6-u$

u:	0	6
v:	6	0

w-plane



Given: $w = z + (2+3i) = x+iy+2+3i$

$u+iv = x+2+i(y+3)$

$\therefore u = x+2$, $v = y+3$

When $x=0$, $u=2$

When $y=0$, $v=3$

When $x+y=1$, $u+v = x+2+y+3 = x+y+5 = 1+5 = 6 \Rightarrow u+v=6$

In the z-plane the line $x=0$ is transformed into $u=2$ in the w-plane.

In the z-plane the line $y=0$ is transformed into $v=3$ in the w-plane.

In the z-plane the line $x+y=1$ is transformed into $u+v=6$ in the w-plane.

(44) Find the analytic function $w = u + iv$ when $v = e^{-2y}(y\cos 2x + x\sin 2x)$ & find u.

Sol: Given $v = e^{-2y}(y\cos 2x + x\sin 2x)$

$$v = ye^{-2y}\cos 2x + e^{-2y}x\sin 2x$$

$$\varphi_2(x,y) = \frac{\partial v}{\partial x} = ye^{-2y}(-2\sin 2x) + e^{-2y}(x.2\cos 2x + \sin 2x)$$

$$\varphi_2(z,0) = 2z\cos 2z + \sin 2z$$

$$\varphi_1(x,y) = \frac{\partial v}{\partial y} = \cos 2x\left(ye^{-2y}.(-2) + e^{-2y}\right) + x\sin 2x \cdot e^{-2y}(-2)$$

$$\varphi_1(z,0) = \cos 2z - 2z\sin 2z$$

By Milne-Thomson method,

$$f(z) = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz$$

$$= \int(\cos 2z - 2z\sin 2z)\,dz + i\int(2z\cos 2z + \sin 2z)\,dz$$

$$= \int \cos 2z\,dz - 2\int z\sin 2z\,dz + i\int 2z\cos 2z\,dz + i\int \sin 2z\,dz$$

$$= \frac{\sin 2z}{2} - 2\left[-z\frac{\cos 2z}{2} + \frac{\sin 2z}{4}\right]$$

$$+ 2i\left[z\frac{\sin 2z}{2} + \frac{\cos 2z}{4}\right] + i\left(-\frac{\cos 2z}{2}\right) + c$$

$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$

$u = z$, $dv = \sin 2z\,dz$

$u' = 1$, $v = -\frac{\cos 2z}{2}$

$v_1 = -\frac{\sin 2z}{4}$

$$= \frac{\sin 2z}{2} + z\cos 2z - \frac{\sin 2z}{2} + i\left(z\sin 2z + \frac{\cos 2z}{2} - \frac{\cos 2z}{2}\right) + c$$

$u = z$, $dv = \cos 2z\,dz$

$u' = 1$, $v = \frac{\sin 2z}{2}$

$v_1 = -\frac{\cos 2z}{4}$

$$\therefore f(z) = z\cos 2z + iz\sin 2z + c$$

$$\cancel{= (x+iy)\cos 2(x+iy) + i(x+iy)\sin 2(x+iy) + c}$$

$$u + iv = z(\cos 2z + i\sin 2z) + c = ze^{i2z} + c = (x+iy)e^{2i(x+iy)} + c$$

$$= (x+iy)e^{i2x-2y} + c = (x+iy)e^{i2x}.e^{-2y} + c$$

$$= e^{-2y}\left[(x+iy)(\cos 2x + i\sin 2x)\right] + c$$

$$= e^{-2y}\left[x\cos 2x + ix\sin 2x + iy\cos 2x - y\sin 2x\right] + c$$

$$= e^{-2y}(x\cos 2x - y\sin 2x) + ie^{-2y}(x\sin 2x + y\cos 2x) + c$$

$$\therefore u = e^{-2y}(x\cos 2x - y\sin 2x)$$

COMPLEX INTEGRATION

Cauchy's integral formula:

If $f(z)$ is analytic inside & on a closed curve c of a simply connected region R & if 'a' is any point with in C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$, the integration around c being taken in the positive direction.

Cauchy's integral formula for derivative:

If a function $f(z)$ is analytic within & on a simple closed curve c & 'a' is any point lying in it, then $f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$.

Note: $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

Cauchy's integral Theorem:

If a function $f(z)$ is analytic & its derivative $f'(z)$ is continuous at all points inside & on a simple closed curve C, then $\int_C f(z)\,dz = 0$.

Defns:

1. A curve which does not cross itself is called a simple closed curve.

2. A curve is called multiple curve if it crosses itself.

3. A region which has no holes is called simply connected region. Otherwise it is said to be multiply connected. →

4. An integral along a simple closed curve is called a contour integral.

---

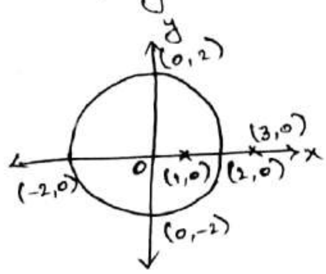
1. Evaluate $\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ where c is the circle $|z|=2$ by using Cauchy's integral formula. [A/M – 2016]

$|z| = 2$

$|x+iy| = 2$

$\sqrt{x^2+y^2} = 2$

$x^2+y^2 = 2^2$



Sol: Consider, $\frac{1}{(z-3)(z-1)} = \frac{A}{z-3} + \frac{B}{z-1}$

$1 = A(z-1) + B(z-3)$

Put $z=1$

$1 = -2B \Rightarrow \boxed{B = -1/2}$

Put $z=3$

$1 = 2A \Rightarrow \boxed{A = 1/2}$

$\therefore \frac{1}{(z-3)(z-1)} = \frac{1/2}{z-3} - \frac{1/2}{z-1}$

$$\therefore \int_c \frac{z+1}{(z-3)(z-1)}\,dz = \frac{1}{2}\int_c \frac{z+1}{z-3}\,dz - \frac{1}{2}\int_c \frac{z+1}{z-1}\,dz$$

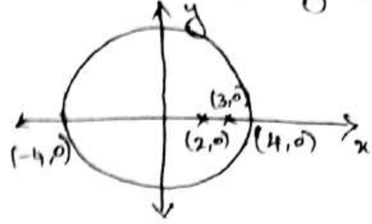
$z=3$ lies outside $|z|=2$ & $z=1$ lies inside $|z|=2$. Here $f(z)=z+1$

Hence by Cauchy's integral formula,

$$\int_c \frac{z+1}{(z-3)(z-1)}\,dz = 0 - \frac{1}{2}\,2\pi i\, f(1) = -\pi i(1+1) = -2\pi i$$

(2) Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}\,dz$ where c is the circle $|z|=4$ by using Cauchy's integral formula. [N/D – 2015]

$|z|=4 \Rightarrow x^2+y^2=4^2$



Sol: Consider, $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$1 = A(z-3) + B(z-2)$

Put $z=3$: $\boxed{1=B}$

Put $z=2$: $1=-A \Rightarrow \boxed{A=-1}$

$$\therefore \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}\,dz = -\int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2}\,dz + \int_C \frac{\sin\pi z^2+\cos\pi z^2}{z-3}\,dz$$

$z=2$ lies inside $|z|=4$ & $z=3$ lies inside $|z|=4$. Here $f(z)=\sin\pi z^2+\cos\pi z^2$

$$\begin{aligned}
\therefore \int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}\,dz &= -2\pi i f(2) + 2\pi i f(3) \\
&= -2\pi i\left[\sin 4\pi + \cos 4\pi\right] + 2\pi i\left[\sin 9\pi + \cos 9\pi\right] \\
&= -2\pi i[0+1] + 2\pi i[\sin\pi + \cos\pi] \\
&= -2\pi i + 2\pi i[0-1] = -2\pi i - 2\pi i = -4\pi i
\end{aligned}$$

(3) Evaluate $\int_c \frac{(z+1)\,dz}{(z-1)(z-2)^2}$ where c is the circle $|z-2|=\frac{1}{2}$ by using Cauchy's integral formula. [N/M – 2017]

$|z-2| = \frac{1}{2} \Rightarrow |x+iy-2| = \frac{1}{2}$

$\sqrt{(x-2)^2+y^2} = \frac{1}{2}$

$(x-2)^2+y^2 = \left(\frac{1}{2}\right)^2$

Sol: Consider, $\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$

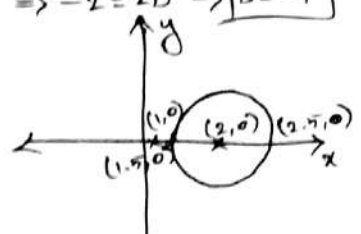
$1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$

Put $z=2$: $\boxed{1=C}$

Put $z=1$: $\boxed{1=A}$

Put $z=0$: $1 = 4A+2B-C \Rightarrow 1 = 4+2B-1 \Rightarrow -2 = 2B \Rightarrow \boxed{B=-1}$

$$\therefore \int_c \frac{(z+1)\,dz}{(z-1)(z-2)^2} = \int_c \frac{(z+1)\,dz}{z-1} - \int_c \frac{(z+1)\,dz}{z-2} + \int_c \frac{(z+1)\,dz}{(z-2)^2}$$



$z=1$ lies outside $|z-2|=1/2$ & $z=2$ lies inside $|z-2|=1/2$. Here $f(z)=z+1$

$f'(z)=1$

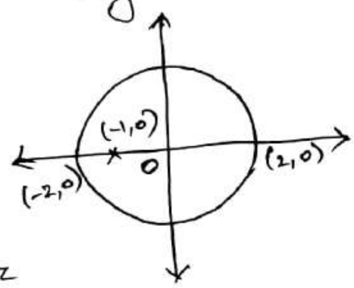
Cauchy's integral formula: $\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$\therefore \int_C \frac{(z+1)\,dz}{(z-1)(z-2)^2} = 0 - 2\pi i f(2) + 2\pi i f'(2) = -2\pi i(2+1) + 2\pi i(1) = -6\pi i + 2\pi i = -4\pi i$

(4) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}dz$ where c is the circle $|z|=2$ by using Cauchy's integral formula. [N/D – 2015]

$|z|=2$

$x^2+y^2=2^2$



Sol: $z=-1$ lies inside $|z|=2$. Here $f(z)=e^{2z}$

Cauchy's integral formula: $\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{n}(a)$.

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!} f^{(3)}(-1) = \frac{2\pi i}{6} f'''(-1)$$

$$= \frac{2\pi i}{6}\left(8e^{2(-1)}\right)$$

$$= \frac{8}{3}\pi i e^{-2}$$

$f(z)=e^{2z}$

$f'(z)=2e^{2z}$

$f''(z)=4e^{2z}$, $f'''(z)=8e^{2z}$

(5) Evaluate $\int_C \frac{z^2}{(z^2+1)^2}dz$ where c is the circle $|z-i|=1$ by using Cauchy's integral formula. [A/M – 2018] [N/D – 2016]

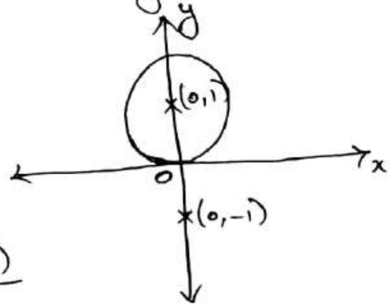
$|z-i|=1$

$|x+iy-i|=1$

$|x+i(y-1)|=1$

$\sqrt{x^2+(y-1)^2}=1$

$x^2+(y-1)^2=1$



Sol: Here $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$

$z=i$ lies inside $|z-i|=1$ & $z=-i$ lies outside $|z-i|=1$.

Here $f(z)=z^2$.

$$\therefore \frac{z^2}{(z^2+1)^2} = \frac{z^2}{[(z+i)(z-i)]^2} = \frac{z^2}{(z+i)^2(z-i)^2} = \frac{\frac{z^2}{(z+i)^2}}{(z-i)^2}$$

$$\int_C \frac{z^2}{(z^2+1)^2}dz = \int_C \frac{\frac{z^2}{(z+i)^2}}{(z-i)^2}dz$$

Here $f(z)=\frac{z^2}{(z+i)^2} \Rightarrow f'(z) = \frac{(z+i)^2.2z - z^2.2(z+i)}{(z+i)^4}$

$$\therefore \int_C \frac{z^2}{(z^2+1)^2}dz = 2\pi i f'(i) = 2\pi i\left[\frac{(i+i)^2 2i - (i)^2 2(i+i)}{(i+i)^4}\right]$$

$$= 2\pi i\left[\frac{(-4)2i + 2(2i)}{(2i)^4}\right] = 2\pi i\left[\frac{-8i+4i}{16}\right] = 2\pi i\left(\frac{-4i}{16}\right) = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}$$

(6) Using Cauchy's integral formula, evaluate $\int \frac{z\,dz}{(z-1)^2(z+2)}$, where c is the circle $|z-1|=1$. [N/D – 2016] [A/M – 2012] [A/M – 2009]

Sol: $z=1$ lies inside $|z-1|=1$ & $z=-2$ lies outside $|z-1|=1$.

$|z-1|=1$

$|x+iy-1|=1$

$\sqrt{(x-1)^2+y^2}=1$

$(x-1)^2+y^2=1$

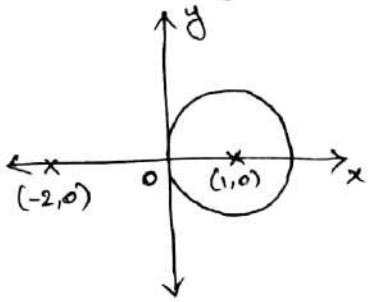
$$\therefore \int_C \frac{z\,dz}{(z-1)^2(z+2)} = \int_C \frac{\frac{z}{z+2}}{(z-1)^2}\,dz \quad \text{Here } f(z)=\frac{z}{z+2}$$

$$\therefore f'(z) = \frac{(z+2)\cdot 1 - z\cdot 1}{(z+2)^2}$$

$$= 2\pi i \cdot f'(1)$$

$$= 2\pi i\left[\frac{(1+2)-1}{(1+2)^2}\right]$$

$$= 2\pi i\left[\frac{2}{9}\right] = \frac{4\pi i}{9}$$



---

(7) Evaluate $\int_C \frac{z}{z^2+1}\,dz$ where c is the circle $|z+i|=1$ by using Cauchy's integral formula. [A/M – 2011]

Sol: Here $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$

$z=i$ lies outside $|z+i|=1$ & $z=-i$ lies inside $|z+i|=1$.

$|z+i|=1$

$|x+iy+i|=1$

$|x+i(y+1)|=1$

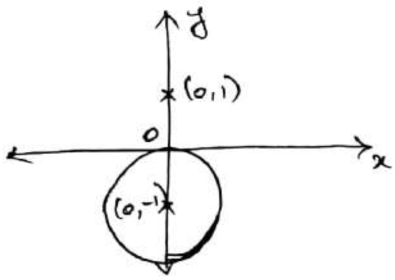
$x^2+(y+1)^2=1$

$$\therefore \int_C \frac{z}{z^2+1}\,dz = \int_C \frac{z}{(z+i)(z-i)}\,dz = \int_C \frac{\frac{z}{z-i}}{z+i}\,dz$$

$$= 2\pi i\, f(-i). \quad \text{Here } f(z)=\frac{z}{z-i}$$

$$= 2\pi i\left(\frac{-i}{-i-i}\right) = 2\pi i\left(\frac{-i}{-2i}\right)$$

$$= 2\pi i\left(\frac{1}{2}\right) = \pi i$$



---

(8) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1+i|=2$ by using Cauchy's integral formula. [N/D – 2012] [N/D – 2011] [N/D – 2010]

Sol: Here $z^2+2z+5=0$

$$z = \frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)} = \frac{-2\pm\sqrt{4-20}}{2}$$

$$= \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$$

$|z+1+i|=2$

$|x+iy+1+i|=2$

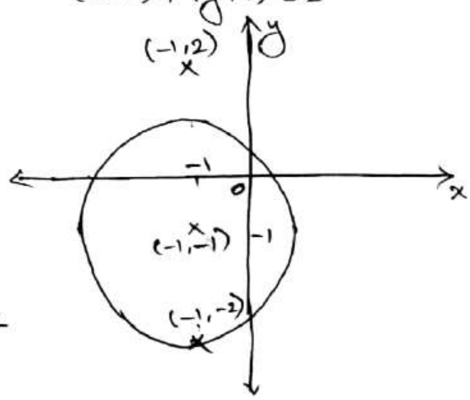
$|x+1+i(y+1)|=2$

$(x+1)^2+(y+1)^2=2^2$

$z=-1+2i$ lies outside $|z+1+i|=2$ &

$z=-1-2i$ lies inside $|z+1+i|=2$.

$$\therefore \int_C \frac{z+4}{(z-(-1+2i))(z-(-1-2i))}\,dz = \int_C \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}\,dz$$



$= 2\pi i\, f(-1-2i)$ Here $f(z) = \dfrac{z+4}{z+1-2i}$

$= 2\pi i \left[\dfrac{-1-2i+4}{-1-2i+1-2i}\right]$

$= 2\pi i \left[\dfrac{3-2i}{-4i}\right] = 2\pi \left(\dfrac{3-2i}{-4}\right) = \dfrac{\pi}{2}(2i-3)$

(9) Evaluate $\displaystyle\int_C \frac{z+1}{(z^2+2z+4)^2}\,dz$ where c is the circle $|z+1+i| = 2$ by using Cauchy's integral formula. [N/D – 2013]

Sol: $z^2+2z+4=0$

$$z = \frac{-2 \pm \sqrt{4-4(1)(4)}}{2} = \frac{-2\pm\sqrt{4-16}}{2}$$

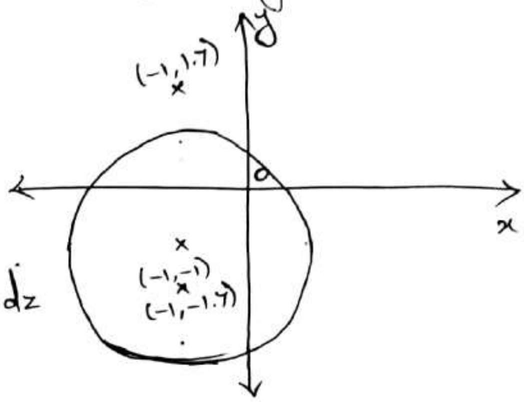
$$= \frac{-2\pm\sqrt{-12}}{2} = \frac{-2\pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

$|z+1+i| = 2$

$|x+iy+1+i| = 2$

$|x+1+i(y+1)| = 2$

$(x+1)^2+(y+1)^2 = 2^2$



$z = -1+i\sqrt{3}$ lies outside $|z+1+i| = 2$ &

$z = -1-i\sqrt{3}$ lies inside $|z+1+i| = 2$.

$$\therefore \int_C \frac{z+1}{(z^2+2z+4)^2}\,dz = \int_C \frac{z+1}{(z-(-1+i\sqrt{3}))^2\,(z-(-1-i\sqrt{3}))^2}\,dz$$

$$= \int_C \frac{\dfrac{z+1}{(z-(-1+i\sqrt{3}))^2}}{(z-(-1-i\sqrt{3}))^2}\,dz$$

Here $f(z) = \dfrac{z+1}{(z+1-i\sqrt{3})^2}$

$$f'(z) = \frac{(z+1-i\sqrt{3})^2(1) - (z+1)\,2(z+1-i\sqrt{3})}{(z+1-i\sqrt{3})^4}$$

$$f'(z) = \frac{(z+1-i\sqrt{3})^2 - 2(z+1)(z+1-i\sqrt{3})}{(z+1-i\sqrt{3})^4}$$

$= 2\pi i\, f'(-1-i\sqrt{3})$

$$= 2\pi i \left[\frac{(-1-i\sqrt{3}+1-i\sqrt{3})^2 - 2(-1-i\sqrt{3}+1)(-1-i\sqrt{3}+1-i\sqrt{3})}{(-1-i\sqrt{3}+1-i\sqrt{3})^4}\right]$$

$$= 2\pi i\left[\frac{(-2i\sqrt{3})^2 - 2(-i\sqrt{3})(-2i\sqrt{3})}{(-2i\sqrt{3})^4}\right] = 2\pi i\left[\frac{-12-12(-1)}{16\times 9}\right]$$

$$= 2\pi i\left[\frac{-12+12}{16\times 9}\right] = 0$$

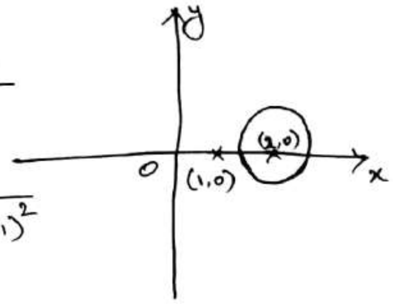
(10) Evaluate $\displaystyle\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where c is the circle $|z-2| = \tfrac{1}{2}$ by using Cauchy's integral formula. [N/D – 2009]

Sol: $z=1$ lies outside $|z-2| = 1/2$ & $z=2$ lies inside $|z-2| = 1/2$.

$|z-2| = 1/2$

$|x+iy-2| = 1/2$

$(x-2)^2 + y^2 = (1/2)^2$



$$\therefore \int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}\,dz$$

Here $f(z) = \dfrac{z}{z-1}$

$$f'(z) = \frac{(z-1)(1) - z(1)}{(z-1)^2} = \frac{z-1-z}{(z-1)^2} = \frac{-1}{(z-1)^2}$$

$$= 2\pi i f'(2)$$

$$= 2\pi i \left(\frac{-1}{(2-1)^2}\right)$$

$$= 2\pi i(-1) = -2\pi i$$

(11) Evaluate $\int_C \dfrac{z+1}{z^2+2z+4}\,dz$ where c is the circle $|z+1+i| = 2$ by using Cauchy's integral formula.

Sol: $z^2+2z+4 = 0 \Rightarrow z = \dfrac{-2 \pm \sqrt{4-4(4)}}{2} = \dfrac{-2 \pm \sqrt{4-16}}{2}$

$$\therefore z = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

$|z+1+i| = 2$

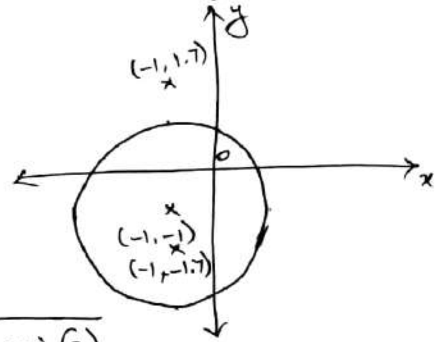
$|x+iy+1+i| = 2$

$|x+1+i(y+1)| = 2$

$(x+1)^2 + (y+1)^2 = 2^2$

$z = -1+i\sqrt{3}$ lies outside $|z+1+i| = 2$ &

$z = -1-i\sqrt{3}$ lies inside $|z+1+i| = 2$.



$$\therefore \int_C \frac{z+1}{z^2+2z+4}\,dz = \int \frac{\frac{z+1}{z-(-1+i\sqrt{3})}}{z-(-1-i\sqrt{3})}\,dz$$

Here $f(z) = \dfrac{z+1}{z-(-1+i\sqrt{3})}$

$$= 2\pi i f(-1-i\sqrt{3})$$

$$= 2\pi i \left[\frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}}\right]$$

$$= 2\pi i \left[\frac{-i\sqrt{3}}{-2i\sqrt{3}}\right] = 2\pi i \left(\frac{1}{2}\right) = \pi i$$

(12) Evaluate $\int_C \dfrac{4-3z}{z(z-1)(z-2)}\,dz$ where c is the circle $|z| = 3/2$ by using Cauchy's integral formula.

Sol: $z=0$ lies inside $|z| = 3/2$,

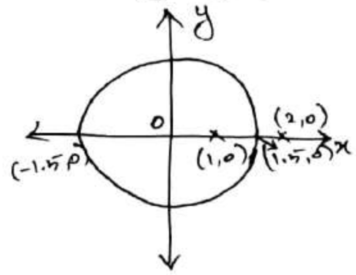
$z=1$ lies inside $|z| = 3/2$ & $z=2$ lies outside $|z| = 3/2$.

$|z| = 3/2$

$|x+iy| = 3/2$

$x^2+y^2 = (3/2)^2 = (1.5)^2$

Consider, $\dfrac{1}{z(z-1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$



$$1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$$

Put $z=1$

$1 = -B \Rightarrow \boxed{B=-1}$

Put $z=2$

$1 = C(2)(1) \rightarrow 2C = 1 \Rightarrow \boxed{C = 1/2}$

Put $z=0$ $\quad 1=A(-1)(-2) \Rightarrow 2A=1 \Rightarrow \boxed{A=\frac{1}{2}}$ $\qquad$ Here $f(z)=4-3z$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz = \frac{1}{2}\int_C \frac{4-3z}{z}dz - \int \frac{4-3z}{z-1}dz + \frac{1}{2}\int \frac{4-3z}{z-2}dz$$

$$= \frac{1}{2}2\pi i f(0) - 2\pi i f(1) + 0$$

$$= \pi i(4-0) - 2\pi i(4-3) = 4\pi i - 2\pi i = 2\pi i$$

(13) Evaluate $\int_C \frac{7z-1}{z^2-3z-4}dz$ where c is the ellipse $x^2+4y^2=4$.

$x^2+4y^2=4$

$\frac{x^2}{4}+\frac{y^2}{1}=1$

$\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$

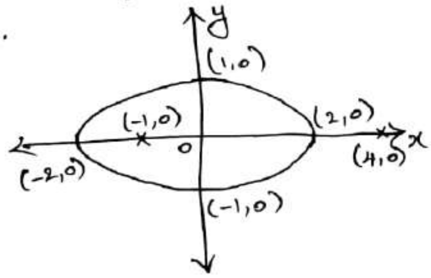
Sol: $z^2-3z-4=0$

×	+
-4	-3
$+1$	-4
$z+1$	$z-4$

$(z+1)(z-4)=0$

$\Rightarrow z=-1, 4$

$z=-1$ lies inside $x^2+4y^2=4$ & $z=4$ lies outside $x^2+4y^2=4$.



$$\int_C \frac{7z-1}{z^2-3z-4}dz = \int_C \frac{7z-1}{(z+1)(z-4)}dz = \int_C \frac{\frac{7z-1}{z-4}}{z+1}dz$$

$$= 2\pi i f(-1) \qquad \text{Here } f(z)=\frac{7z-1}{z-4}$$

$$= 2\pi i\left[\frac{-7-1}{-5}\right] = 2\pi i\left(\frac{-8}{-5}\right) = \frac{16\pi i}{5}$$

Cauchy's residue theorem:

If $f(z)$ be analytic at all points inside & on a simple closed curve c, except for a finite number of isolated singularities $z_1, z_2, \dots, z_n$ inside c, then

$\int_C f(z)dz = 2\pi i$[sum of the residues of $f(z)$ at $z_1, z_2, \dots, z_n$]

(14) Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$ where c is the circle $|z|=3$ using Cauchy's residue theorem. [N/D-2011] [A/M-2013]

$$\left[\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right.$$

$\left.\text{where } a \text{ is the pole of order } m.\right]$

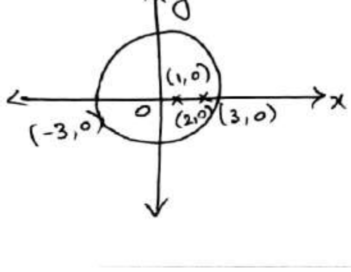
Sol: Here $f(z)=\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$

$z=1$ lies inside $|z|=3$ & $z=2$ lies inside $|z|=3$.

$z=1$ is a pole of order 1 & $z=2$ is a pole of order 1.

$|z|=3 \Rightarrow |x+iy|=3$

$x^2+y^2=3^2$



$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$$

$$= \frac{\sin\pi+\cos\pi}{1-2} = \frac{0-1}{-1} = 1$$

$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} (z-2)\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \frac{\sin 4\pi + \cos 4\pi}{2-1} = \frac{0+1}{1} = 1$$

$$\therefore \int_C f(z)\,dz = 2\pi i \times \text{sum of the residues} = 2\pi i \times (1+1) = 4\pi i$$

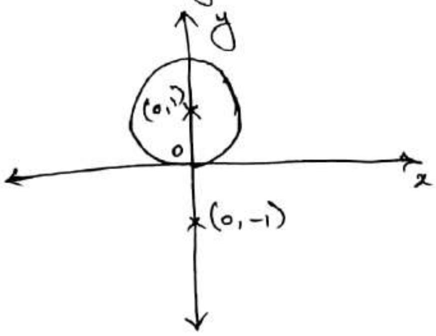
(15) Evaluate $\int_C \frac{z\,dz}{(z^2+1)^2}$, where c is the circle $|z-i|=1$ using Cauchy's residue theorem. [N/D – 2016]

$|z-i|=1$

$|x+iy-i|=1$

$|x+i(y-1)|=1$

$x^2+(y-1)^2=1$



Sol: $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$

$z=i$ lies inside $|z-i|=1$ & $z=-i$ lies outside $|z-i|=1$.

$\therefore z=i$ is a pole of order 2. $(a=i, m=2)$

$$\int_C \frac{z\,dz}{(z+i)^2(z-i)^2} = \int_C \frac{z\,dz}{(z^2+1)^2}$$

Here $f(z) = \dfrac{z}{(z+i)^2(z-i)^2}$

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{z}{(z+i)^2(z-i)^2}\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{z}{(z+i)^2}\right] = \lim_{z\to i}\frac{(z+i)^2\cdot 1 - z\,2(z+i)}{(z+i)^4}$$

$$= \frac{(2i)^2 - 2i(2i)}{(2i)^4} = \frac{-4+4}{16} = 0$$

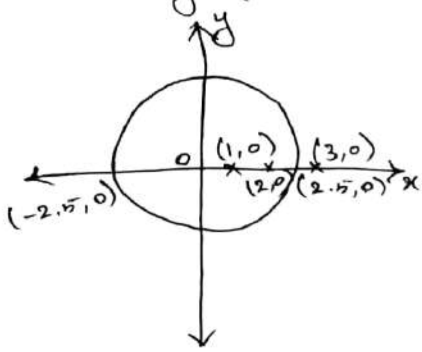
$$\therefore \int_C \frac{z\,dz}{(z^2+1)^2} = 2\pi i \times (\text{Sum of the residues}) = 2\pi i \times (0) = 0$$

(16) Evaluate $\int_C \frac{z^3\,dz}{(z-1)^4(z-2)(z-3)}$ where c is the circle $|z|=2.5$ using Cauchy's residue theorem. [N/D – 2015]

$|z|=2.5$

$|x+iy|=2.5$

$x^2+y^2=(2.5)^2$



Sol: $z=1$ lies inside $|z|=2.5$, $z=2$ lies inside $|z|=2.5$ & $z=3$ lies outside $|z|=2.5$.

$z=1$ is a pole of order 4. $(a=1, m=4)$

$z=2$ is a pole of order 1. $(a=2, m=1)$

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$\underset{z=1}{\text{Res}}\, f(z) = \frac{1}{3!}\lim_{z\to 1}\frac{d^3}{dz^3}\left[(z-1)^4\frac{z^3}{(z-1)^4(z-2)(z-3)}\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left[\frac{z^3}{z^2-5z+6}\right] = \frac{1}{6}\lim_{z\to 1}\frac{d^3}{dz^3}\left[z^3(z^2-5z+6)^{-1}\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^2}{dz^2}\left[z^3(-1)(z^2-5z+6)^{-2}(2z-5)+(z^2-5z+6)^{-1}3z^2\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(-2z^4+5z^3)(z^2-5z+6)^{-2}+3z^2(z^2-5z+6)^{-1}\right]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d}{dz}\Big[(-2z^4+5z^3)(-2)(z^2-5z+6)^{-3}(2z-5)+(z^2-5z+6)^{-2}(-8z^3+15z^2)$$
$$+3z^2(-1)(z^2-5z+6)^{-2}(2z-5)+(z^2-5z+6)^{-1}6z\Big]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d}{dz}\Big[(4z^4-10z^3)(2z-5)(z^2-5z+6)^{-3}+(-8z^3+15z^2)(z^2-5z+6)^{-2}$$
$$+(-6z^3+15z^2)(z^2-5z+6)^{-2}+6z(z^2-5z+6)^{-1}\Big]$$

$$= \frac{1}{6}\lim_{z\to 1}\frac{d}{dz}\Big[(8z^5-40z^4+50z^3)(z^2-5z+6)^{-3}+(-8z^3+15z^2)(z^2-5z+6)^{-2}$$
$$+(-6z^3+15z^2)(z^2-5z+6)^{-2}+6z(z^2-5z+6)^{-1}\Big]$$

$$= \frac{1}{6}\lim_{z\to 1}\Big[(8z^5-40z^4+50z^3)(-3)(z^2-5z+6)^{-4}(2z-5)+(z^2-5z+6)^{-3}(40z^4-160z^3+150z^2)$$
$$+(-8z^3+15z^2)(-2)(z^2-5z+6)^{-3}(2z-5)+(z^2-5z+6)^{-2}(-24z^2+30z)$$
$$+(-6z^3+15z^2)(-2)(z^2-5z+6)^{-3}(2z-5)+(z^2-5z+6)^{-2}(-18z^2+30z)$$
$$+6z(-1)(z^2-5z+6)^{-2}(2z-5)+(z^2-5z+6)^{-1}\cdot 6\Big]$$

$$= \frac{1}{6}\Big[(18)(-3)(2)^{-4}(-3)+(2)^{-3}(30)+(7)(-2)(2)^{-3}(-3)+(2)^{-2}(6)$$
$$+(9)(-2)(2)^{-3}(-3)+(2)^{-2}(12)+6(-1)(2)^{-2}(-3)+(2)^{-1}\cdot 6\Big]$$

$$= \frac{1}{6}\left[\frac{18\times 9}{2^4}+\frac{30}{2^3}+\frac{7\times 6}{2^3}+\frac{6}{2^2}+\frac{9\times 6}{2^3}+\frac{12}{2^2}+\frac{18}{2^2}+\frac{6}{2}\right]$$

$$= \frac{1}{6}\left[\frac{81}{8}+\frac{15}{4}+\frac{21}{4}+\frac{3}{2}+\frac{27}{4}+3+\frac{9}{2}+3\right] = \frac{1}{6}\left[\frac{81}{8}+\frac{63}{4}+\frac{12}{2}+6\right]$$

$$= \frac{1}{6}\left[\frac{81}{8}+\frac{63}{4}+12\right] = \frac{1}{6}\left[\frac{81+126+96}{8}\right] = \frac{1}{6}\left(\frac{303}{8}\right) = \frac{101}{2\times 8} = \frac{101}{16}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2}\left[(z-2)\frac{z^3}{(z-1)^4(z-2)(z-3)}\right]$$

$$= \lim_{z\to 2}\left[\frac{z^3}{(z-1)^4(z-3)}\right] = \frac{8}{1\times -1} = -8$$

$$\therefore \int_C \frac{z^3\,dz}{(z-1)^4(z-2)(z-3)} = 2\pi i \times \text{Sum of the residues}$$

$$= 2\pi i \times\left(\frac{101}{16}-8\right) = 2\pi i\times\left(\frac{101-128}{16}\right) = 2\pi i\times\frac{-27}{16}$$

$$= \frac{-27}{8}\pi i$$

(17) Evaluate $\int_C \frac{z-1}{(z-1)^2(z-2)}\,dz$ where c is the circle $|z-i|=2$ using Cauchy's residue theorem. [A/M-2012] [A/M-2014]

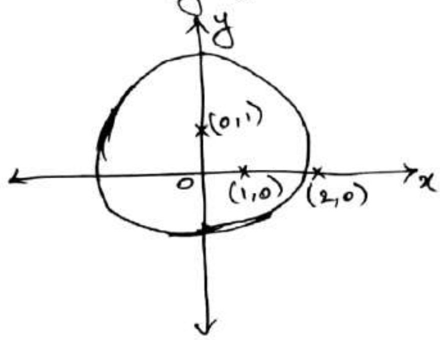
Sol: Given $\int_C \frac{z-1}{(z-1)^2(z-2)}\,dz = \int_C \frac{1}{(z-1)(z-2)}\,dz$

$|z-i|=2$

$|x+iy-i|=2$

$|x+i(y-1)|=2$

$x^2+(y-1)^2=2^2$



$z=1$ lies inside $|z-i|=2$ & $z=2$ lies outside $|z-i|=2$.

$z=1$ is a pole of order 1 $(a=1, m=1)$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-2)} = \lim_{z\to 1}\frac{1}{z-2} = -1$$

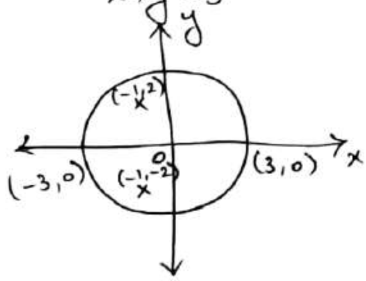
$$\therefore \int_C \frac{z-1}{(z-1)^2(z-2)}\,dz = 2\pi i \times \text{Sum of the residues}$$

$$= 2\pi i \times -1 = -2\pi i$$

(18) Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$ where c is the circle $|z|=3$ using Cauchy's residue theorem. [N/D-2015]

$|z|=3 \Rightarrow |x+iy|=3$

$x^2+y^2=3^2$



Sol: $z^2+2z+5=0$

$$z = \frac{-2\pm\sqrt{4-4(5)}}{2} = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm\sqrt{-16}}{2}$$

$$\therefore z = \frac{-2\pm 4i}{2} = -1\pm 2i$$

$z=-1+2i$ lies inside $|z|=3$ & $z=-1-2i$ lies inside $|z|=3$.

$z=-1+2i$ is a pole of order 1. $(a=-1+2i, m=1)$

$z=-1-2i$ is a pole of order 1. $(a=-1-2i, m=1)$

$$\operatorname*{Res}_{z=-1+2i} f(z) = \lim_{z\to -1+2i}(z-(-1+2i))\frac{z-3}{(z-(-1+2i))(z-(-1-2i))}$$

$$= \frac{-1+2i-3}{-1+2i+1+2i} = \frac{2i-4}{4i} = \frac{(2i-4)(-i)}{4i(-i)} = \frac{2+4i}{4} = \frac{1+2i}{2}$$

$$\operatorname*{Res}_{z=-1-2i} f(z) = \lim_{z\to -1-2i}(z-(-1-2i))\frac{z-3}{(z-(-1+2i))(z-(-1-2i))}$$

$$= \frac{-1-2i-3}{-1-2i+1-2i} = \frac{-2i-4}{-4i}\times\frac{i}{i} = \frac{2-4i}{4} = \frac{1-2i}{2}$$

$$\therefore \int_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i \times \text{Sum of the residues} = 2\pi i \times\left(\frac{1+2i}{2}+\frac{1-2i}{2}\right)$$

$$= 2\pi i \times \left(\frac{2}{2}\right) = 2\pi i$$

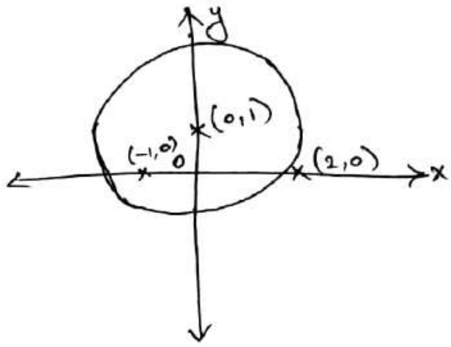
(19) Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ where c is the circle $|z-i|=2$ using Cauchy's residue theorem.

$|z-i|=2$

$|x+iy-i|=2$

$|x+i(y-1)|=2$

$x^2+(y-1)^2=2^2$



Sol: $z=-1$ lies inside $|z-i|=2$ & $z=2$ lies outside $|z-i|=2$.

$z=-1$ is a pole of order 2. $(a=-1, m=2)$

$$\underset{z=-1}{\text{Res}}\, f(z) = \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z-2}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$$

$$= \frac{-3+2}{(-3)^2} = \frac{-1}{9}$$

$$\therefore \int_c \frac{z-1}{(z+1)^2(z-2)}dz = 2\pi i \times \text{Sum of the residues} = 2\pi i \times \frac{-1}{9} = -\frac{2\pi i}{9}$$

Taylor's series: Taylor's series about $z=a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \text{ to } \infty.$$

Laurent's series: $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$

where $a_n = \frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-a)^{n+1}}dz$ & $b_n = \frac{1}{2\pi i}\int_{c_2}\frac{f(z)}{(z-a)^{-n+1}}dz$.

---

(20) Expand $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in Laurent's series valid for $1<|z+1|<3$. [N/D – 2015] [A/M – 2010]

Sol: Given $f(z) = \frac{7z-2}{(z+1)z(z-2)} = \frac{A}{z+1} + \frac{B}{z} + \frac{C}{z-2}$

$7z-2 = Az(z-2) + B(z+1)(z-2) + Cz(z+1)$

Put $z=2$: $12 = 6C \Rightarrow \boxed{C=2}$

Put $z=-1$: $-9 = 3A \Rightarrow \boxed{A=-3}$

Put $z=0$: $-2 = -2B \Rightarrow \boxed{B=1}$

$$\therefore f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2}$$

Given: $1<|z+1|<3$

Let $u=z+1 \Rightarrow z=u-1$

$1<|u|<3$ (ie) $1<|u|$ & $|u|<3$

$\left|\frac{1}{u}\right|<1$ & $\left|\frac{u}{3}\right|<1$

$$\therefore f(z) = \frac{-3}{u-1+1} + \frac{1}{u-1} + \frac{2}{u-1-2} = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$$= \frac{-3}{u} + \frac{1}{u(1-1/u)} + \frac{2}{-3(1-u/3)} = \frac{-3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{-3}{u} + \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\cdots\right] - \frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right]$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\cdots\right] - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$= \frac{-3}{z+1} + \left(\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right) - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$= \frac{-3}{z+1} + \frac{1}{z+1} + \sum_{n=2}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

$$= \frac{-2}{z+1} + \sum_{n=2}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n}$$

(21) Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in Laurent's series valid for (i) $2<|z|<3$ & (ii) $|z|>3$. [N/D-2011] [A/M-2014], [A/M-2011] [A/M-2013] [A/M-2009]

Sol: Given $f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$

$$z^2+5z+6 \,\big|\, z^2-1 \,\big(1$$
$$z^2+5z+6$$
$$(-)(-)\;(-)$$
$$-5z-7$$

$$= 1 + \frac{-5z-7}{z^2+5z+6} = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

Consider, $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$$-5z-7 = A(z+3) + B(z+2)$$

Put $z=-3$

$15-7 = -B \Rightarrow \boxed{B=-8}$

Put $z=-2$

$10-7 = A \Rightarrow \boxed{A=3}$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) Given: $2<|z|<3$ (ie) $2<|z|$ & $|z|<3$ (ie) $\left|\frac{2}{z}\right|<1$ & $\left|\frac{z}{3}\right|<1$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(z/3+1)} = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+z/3\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\cdots\right) - \frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\cdots\right)$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) Given: $|z|>3$ (ie) $\left|\frac{3}{z}\right|<1 \rightarrow \left|\frac{2}{z}\right|<1$

$$\therefore f(z) = 1+\frac{3}{z\left(1+2/z\right)} - \frac{8}{z\left(1+3/z\right)} = 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2 - \cdots\right) - \frac{8}{z}\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2 - \cdots\right)$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

---

22 Expand $f(z) = \frac{1}{z^2+4z+3}$ in Laurent's series valid in the regions (i) $|z|<1$ [A/M-2017]

(ii) $0<|z+1|<2$ (iii) $|z|>3$ (iv) $1<|z|<3$.

×	+
3	4
1	3
$z+1$	$z+3$

Sol: Given $f(z) = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$

$$= \frac{A}{z+1} + \frac{B}{z+3}$$

$$\therefore 1 = A(z+3) + B(z+1)$$

Put $z=-3$

$1=-2B \Rightarrow \boxed{B=-1/2}$

Put $z=-1$

$1=2A \Rightarrow \boxed{A=1/2}$

$$\therefore f(z) = \frac{1/2}{z+1} - \frac{1/2}{z+3} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}$$

(i) Given: $|z|<1 \Rightarrow \left|\frac{z}{3}\right|<1$

$$\therefore f(z) = \frac{1}{2}\frac{1}{1+z} - \frac{1}{2}\frac{1}{3\left(1+z/3\right)} = \frac{1}{2}(1+z)^{-1} - \frac{1}{2}\frac{1}{3}\left(1+z/3\right)^{-1}$$

$$= \frac{1}{2}\left[1-z+z^2 - \cdots\right] - \frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) Given: $0<|z+1|<2$

Put $u=z+1 \Rightarrow z=u-1$

$\therefore 0<|u|<2$ (ie) $0<|u|$ & $|u|<2 \Rightarrow \left|\frac{u}{2}\right|<1$

$$\therefore f(z) = \frac{1}{2}\frac{1}{u} - \frac{1}{2}\frac{1}{u-1+3} = \frac{1}{2}\frac{1}{u} - \frac{1}{2}\frac{1}{u+2} = \frac{1}{2}\frac{1}{u} - \frac{1}{2}\frac{1}{2\left(1+u/2\right)}$$

$$= \frac{1}{2}\frac{1}{u} - \frac{1}{2}\frac{1}{2}\left(1+\frac{u}{2}\right)^{-1} = \frac{1}{2u} - \frac{1}{4}\left(1-\frac{u}{2}+\left(\frac{u}{2}\right)^2 - \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+1}{2}\right)^n$$

(iii) Given: $|z|>3 \Rightarrow \left|\frac{3}{z}\right|<1$

$$\therefore f(z)=\frac{1}{2}\frac{1}{1+z}-\frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}=\frac{1}{2}(1+z)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=\frac{1}{2}\left(1-z+z^2-\dots\right)-\frac{1}{2z}\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\dots\right)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n-\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

(iv) Given: $1<|z|<3$ (ie) $1<|z|$ & $|z|<3 \Rightarrow \left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{3}\right|<1$

$$\therefore f(z)=\frac{1}{2}\frac{1}{1+z}-\frac{1}{2}\frac{1}{z+3}=\frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2}\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\left(1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\dots\right)-\frac{1}{6}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\dots\right)$$

$$=\frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n-\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(23) Expand $f(z)=\dfrac{6z+5}{(z+1)z(z-2)}$ in Laurent's series valid for $1<|z+1|<3$. [A/M-2018]

Sol: Given $f(z)=\dfrac{6z+5}{(z+1)z(z-2)}=\dfrac{A}{z+1}+\dfrac{B}{z}+\dfrac{C}{z-2}$

$$6z+5=Az(z-2)+B(z+1)(z-2)+Cz(z+1)$$

Put $z=0$: $5=-2B \Rightarrow \boxed{B=-5/2}$

Put $z=2$: $17=C(2)(3) \Rightarrow \boxed{C=17/6}$

Put $z=-1$: $-1=A(-1)(-3) \Rightarrow \boxed{A=-1/3}$

$$\therefore f(z)=\frac{-1/3}{z+1}-\frac{5/2}{z}+\frac{17/6}{z-2}$$

Given: $1<|z+1|<3$. Let $u=z+1 \Rightarrow z=u-1$

$\therefore 1<|u|<3$ (ie) $1<|u|$ & $|u|<3 \Rightarrow \left|\frac{1}{u}\right|<1$ & $\left|\frac{u}{3}\right|<1$

$$\therefore f(z)=-\frac{1}{3}\frac{1}{1+z}-\frac{5}{2}\frac{1}{z}+\frac{17}{6}\frac{1}{z-2}=-\frac{1}{3}\frac{1}{u}-\frac{5}{2}\frac{1}{u-1}+\frac{17}{6}\frac{1}{u-3}$$

$$=-\frac{1}{3}\frac{1}{u}-\frac{5}{2}\frac{1}{u\left(1-\frac{1}{u}\right)}+\frac{17}{6}\frac{1}{3\left(\frac{u}{3}-1\right)}$$

$$=-\frac{1}{3}\frac{1}{u}-\frac{5}{2u}\left(1-\frac{1}{u}\right)^{-1}-\frac{17}{6}\frac{1}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$=\frac{-1}{3u}-\frac{5}{2u}\left(1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\dots\right)-\frac{17}{18}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\dots\right)$$

$$=\frac{-1}{3u}-\frac{5}{2u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n-\frac{17}{8}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n$$

$$\therefore f(z) = \frac{-1}{3(z+1)} - \frac{5}{2(z+1)} \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^n - \frac{17}{8} \sum_{n=0}^{\infty} \left(\frac{z+1}{3}\right)^n$$

24) Expand $f(z) = \dfrac{z^2-4z+2}{z^3-2z^2-5z+6}$ in Laurent's series valid for $3 < |z+2| < 5$. [N/D – 2015]

Sol: Given $f(z) = \dfrac{z^2-4z+2}{z^3-2z^2-5z+6}$

$$= \frac{z^2-4z+2}{(z-1)(z-3)(z+2)}$$

$$= \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$$

$z=1$	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$z^2 - z - 6 = 0$

$(z-3)(z+2) = 0$

$\therefore z^2-4z+2 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$

Put $z=3$

$9-12+2 = B(2)(5)$

$-1 = 10B \Rightarrow \boxed{B = -1/10}$

Put $z=-2$

$4+8+2 = C(-3)(-5)$

$14 = 15C \Rightarrow \boxed{C = 14/15}$

Put $z=1$

$1-4+2 = A(-2)(3)$

$-1 = -6A \Rightarrow \boxed{A = 1/6}$

$$\therefore f(z) = \frac{1/6}{z-1} - \frac{1/10}{z-3} + \frac{14/15}{z+2} = \frac{1}{6}\frac{1}{z-1} - \frac{1}{10}\frac{1}{z-3} + \frac{14}{15}\frac{1}{z+2}$$

Given: $3 < |z+2| < 5$

Let $u = z+2 \Rightarrow z = u-2$

$\therefore 3 < |u| < 5$ (i) $3 < |u|$ & $|u| < 5 \Rightarrow \left|\frac{3}{u}\right| < 1$ & $\left|\frac{u}{5}\right| < 1$

$$\therefore f(z) = \frac{1}{6}\frac{1}{u-2-1} - \frac{1}{10}\frac{1}{u-2-3} + \frac{14}{15}\frac{1}{u}$$

$$= \frac{1}{6}\frac{1}{u-3} - \frac{1}{10}\frac{1}{u-5} + \frac{14}{15}\frac{1}{u}$$

$$= \frac{1}{6}\frac{1}{u\left(1-3/u\right)} - \frac{1}{10}\frac{1}{5\left(u/5-1\right)} + \frac{14}{15u}$$

$$= \frac{1}{6u}\left(1-\frac{3}{u}\right)^{-1} + \frac{1}{50}\left(1-\frac{u}{5}\right)^{-1} + \frac{14}{15u}$$

$$= \frac{1}{6u}\left(1+\frac{3}{u}+\left(\frac{3}{u}\right)^2+\cdots\right) + \frac{1}{50}\left(1+\frac{u}{5}+\left(\frac{u}{5}\right)^2+\cdots\right) + \frac{14}{15u}$$

$$= \frac{1}{6u}\sum_{n=0}^{\infty}\left(\frac{3}{u}\right)^n + \frac{1}{50}\sum_{n=0}^{\infty}\left(\frac{u}{5}\right)^n + \frac{14}{15u}$$

$$= \frac{1}{6(z+2)}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{50}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{14}{15(z+2)}$$

(25) Expand $f(z)=\dfrac{7z-2}{(z-2)(z+1)}$ in Laurent's series valid for $|z+1|<1$ & $|z+1|>3$.
[A/M – 2014]

Sol: Given $f(z)=\dfrac{7z-2}{(z-2)(z+1)}=\dfrac{A}{z-2}+\dfrac{B}{z+1}$

$$7z-2=A(z+1)+B(z-2)$$

Put $z=-1$

$-7-2=-3B \Rightarrow -9=-3B \Rightarrow \boxed{B=3}$

Put $z=2$

$14-2=3A \Rightarrow 12=3A \Rightarrow \boxed{A=4}$

$$\therefore f(z)=\frac{4}{z-2}+\frac{3}{z+1}$$

Given: $|z+1|<1$. Let $u=z+1 \Rightarrow z=u-1$

$|u|<1 \Rightarrow \left|\frac{u}{3}\right|<1$

$$\therefore f(z)=\frac{4}{u-1-2}+\frac{3}{u}=\frac{4}{u-3}+\frac{3}{u}=\frac{4}{3\left(u/3-1\right)}+\frac{3}{u}$$

$$=\frac{-4}{3}\left(1-\frac{u}{3}\right)^{-1}+\frac{3}{u}=\frac{-4}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots\right)+\frac{3}{u}$$

$$=\frac{-4}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n+\frac{3}{u}=-\frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n+\frac{3}{z+1}$$

Given: $|z+1|>3$. Let $z+1=u \Rightarrow z=u-1$

$|u|>3 \Rightarrow \left|\frac{3}{u}\right|<1$

$$\therefore f(z)=\frac{4}{u-3}+\frac{3}{u}=\frac{4}{u\left(1-3/u\right)}+\frac{3}{u}=\frac{4}{u}\left(1-\frac{3}{u}\right)^{-1}+\frac{3}{u}$$

$$=\frac{4}{u}\left(1+\frac{3}{u}+\left(\frac{3}{u}\right)^2+\cdots\right)+\frac{3}{u}=\frac{4}{u}\sum_{n=0}^{\infty}\left(\frac{3}{u}\right)^n+\frac{3}{u}$$

$$=\frac{4}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n+\frac{3}{z+1}$$

(26) Expand $f(z)=\dfrac{3z-2}{z(z^2-4)}$ in Laurent's series valid for $2<|z+2|<4$. [N/D – 2014]

Sol: Given $f(z)=\dfrac{3z-2}{z(z^2-4)}=\dfrac{3z-2}{z(z^2-2^2)}=\dfrac{3z-2}{z(z+2)(z-2)}=\dfrac{A}{z}+\dfrac{B}{z+2}+\dfrac{C}{z-2}$

$$3z-2=A(z+2)(z-2)+Bz(z-2)+Cz(z+2)$$

Put $z=-2$

$-6-2=B(-2)(-4)$

$-8=8B \Rightarrow \boxed{B=-1}$

Put $z=2$

$6-2=C(2)(4)$

$4=8C \Rightarrow \boxed{C=1/2}$

Put $z=0$

$-2=A(-4) \Rightarrow \boxed{A=1/2}$

$$\therefore f(z)=\frac{1}{2}\frac{1}{z}-\frac{1}{z+2}+\frac{1}{2}\frac{1}{z-2}$$

Given: $2<|z+2|<4$ Put $u=z+2 \Rightarrow z=u-2$

$2<|u|<4$ (ie) $2<|u|$ & $|u|<4 \Rightarrow \left|\frac{2}{u}\right|<1$ & $\left|\frac{u}{4}\right|<1$

$$\therefore f(z) = \frac{1}{2}\frac{1}{u-2} - \frac{1}{u} + \frac{1}{2}\frac{1}{u-4} = \frac{1}{2}\frac{1}{u\left(1-2/u\right)} - \frac{1}{u} + \frac{1}{2}\frac{1}{4\left(u/4 - 1\right)}$$

$$= \frac{1}{2u}\left(1-\frac{2}{u}\right)^{-1} - \frac{1}{u} - \frac{1}{8}\left(1-\frac{u}{4}\right)^{-1}$$

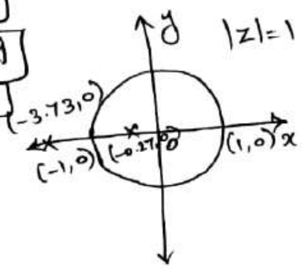
$$= \frac{1}{2u}\left(1+\frac{2}{u}+\left(\frac{2}{u}\right)^2+\cdots\right) - \frac{1}{u} - \frac{1}{8}\left(1+\frac{u}{4}+\left(\frac{u}{4}\right)^2+\cdots\right)$$

$$= \frac{1}{2u}\sum_{n=0}^{\infty}\left(\frac{2}{u}\right)^n - \frac{1}{u} - \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{u}{4}\right)^n$$

$$= \frac{1}{2(z+2)}\sum_{n=0}^{\infty}\left(\frac{2}{z+2}\right)^n - \frac{1}{z+2} - \frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{z+2}{4}\right)^n$$

CONTOUR INTEGRATION:

(27) Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ using contour integration. [NID – 2010] [NID – 2009] [ARM – 2010]



Sol: Let $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \int_C \frac{\frac{1}{iz}dz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \quad \text{where } c \text{ is } |z|=1$$

$$= \frac{1}{i}\int_C \frac{\frac{dz}{z}}{2+\frac{1}{2}\left(\frac{z^2+1}{z}\right)} = \frac{2}{i}\int_C \frac{dz}{z\left(\frac{4z+z^2+1}{z}\right)} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$$

$$z^2+4z+1 = 0$$

$$z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = \frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}$$

Let $\alpha = -2+\sqrt{3} = -2+1.73 = -0.27$ is a simple pole which lies inside c.

$\beta = -2-\sqrt{3} = -2-1.73 = -3.73$ is a simple pole which lies outside c.

$$\therefore \int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C\frac{dz}{(z-\alpha)(z-\beta)} \qquad \text{Here } f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{-2+\sqrt{3}-(-2-\sqrt{3})} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$\therefore \int_C\frac{dz}{(z-\alpha)(z-\beta)} = 2\pi i \times \text{Sum of the residues} = 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \frac{2}{i}\times\frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Scanned by CamScanner

(28) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ using contour integration. [A/M – 2018] [N/D – 2013]

Sol: Let $z = e^{i\theta}$ $\qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$

$dz = ie^{i\theta} d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$

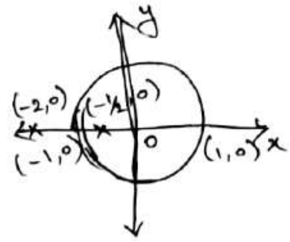
$z^2 = \left(e^{i\theta}\right)^2 = e^{i2\theta} = \cos 2\theta + i \sin 2\theta$

Real part of $z^2 = \cos 2\theta$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = R.P. \int_C \frac{z^2}{5+4 \times \frac{1}{2}\left(\frac{z^2+1}{z}\right)} \frac{1}{iz} dz \quad \text{where } c \text{ is } |z|=1$$

$$= R.P. \int_C \frac{z^2}{5+2\left(\frac{z^2+1}{z}\right)} \frac{1}{iz} dz = R.P. \int_C \frac{z^2 \cdot z}{5z+2z^2+2} \frac{1}{iz} dz$$

$$= R.P. \frac{1}{i} \int_C \frac{z^2}{2z^2+5z+2} dz = R.P. \frac{1}{2i} \int_C \frac{z^2}{z^2+\frac{5}{2}z+1} dz$$



Consider, $z^2 + \frac{5}{2}z + 1 = 0$

$$z = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{-\frac{5}{2} \pm \sqrt{9/4}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{-5 \pm 3}{4} = \frac{-5+3}{4}, \frac{-5-3}{4}$$

$\therefore z = \frac{-1}{2}, -2$

Let $\alpha = \frac{-1}{2}$ is a simple pole which lies inside c.

$\beta = -2$ is a simple pole which lies outside c.

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = R.P. \frac{1}{2i} \int \frac{z^2}{(z-\alpha)(z-\beta)} dz$$

$= R.P. \frac{1}{2i} \times 2\pi i \times$ Sum of the residues

$= R.P. \pi \times \frac{1}{6} = \frac{\pi}{6}$

Here $f(z) = \frac{z^2}{(z-\alpha)(z-\beta)}$

$$\underset{z=\alpha}{\text{Res}} f(z) = \lim_{z \to \alpha} (z-\alpha) \frac{z^2}{(z-\alpha)(z-\beta)}$$

$$= \frac{\alpha^2}{\alpha - \beta} = \frac{(-1/2)^2}{-1/2+2} = \frac{1/4}{3/2}$$

$$= \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

(29) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a > b > 0$ using contour integration.

Sol: Let $z = e^{i\theta}$

$dz = ie^{i\theta} d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$

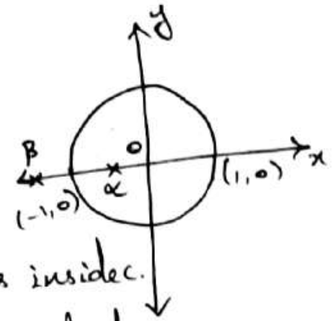
$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{dz/iz}{a+b/2\left(\frac{z^2+1}{z}\right)} = \int_C \frac{dz/iz \times 2z}{2az+bz^2+b} = \frac{2}{i} \int_C \frac{dz}{bz^2+2az+b} \quad \text{where } c \text{ is } |z|=1$$

$$= \frac{2}{bi}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

Consider $z^2+\frac{2a}{b}z+1=0$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2} = \frac{-\frac{2a}{b} \pm \frac{2}{b}\sqrt{a^2-b^2}}{2}$$

$$\therefore z = \frac{-a}{b} \pm \frac{\sqrt{a^2-b^2}}{b} = \frac{1}{b}\left[-a \pm \sqrt{a^2-b^2}\right]$$



Let $\alpha = \frac{1}{b}\left(-a+\sqrt{a^2-b^2}\right)$ & $\beta = \frac{1}{b}\left(-a-\sqrt{a^2-b^2}\right)$

$(\sqrt{3}=1.732)$

Given: $a>b>0$. Let $b=1,\ a=2$

$\therefore \alpha = -2+\sqrt{4-1} = -2+\sqrt{3} = -0.268$ is a simple pole which lies inside c.

$\beta = -2-\sqrt{4-1} = -2-\sqrt{3} = -3.732$ is a simple pole which lies outside c.

Here $f(z) = \frac{1}{z^2+\frac{2a}{b}z+1} = \frac{1}{(z-\alpha)(z-\beta)}$

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\frac{1}{b}\left(-a+\sqrt{a^2-b^2}\right)-\frac{1}{b}\left(-a-\sqrt{a^2-b^2}\right)} = \frac{b}{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}}$$

$$= \frac{b}{2\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{bi}\times 2\pi i \times \text{Sum of the residues}$$

$$= \frac{2}{bi}\times 2\pi i \times \frac{b}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

---

(30) Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$, $a>b>0$ using contour integration.

$z^2 = (e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

Sol: Let $z = e^{i\theta}$

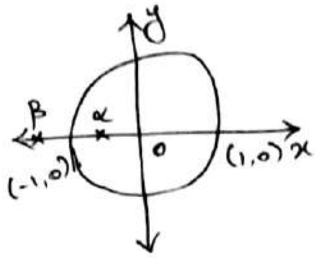
$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = R.P\left(\frac{1-e^{i2\theta}}{2}\right) = R.P\left(\frac{1-z^2}{2}\right)\ ; \ \cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = R.P\int_C \frac{\frac{1-z^2}{2}}{a+b\frac{1}{2}\left(\frac{z^2+1}{z}\right)}\frac{dz}{iz} \quad \text{where } c \text{ is } |z|=1$$

$$= R.P\int_C \frac{\frac{1-z^2}{2}}{\frac{2az+bz^2+b}{2z}}\frac{dz}{iz} = R.P\int_C \frac{1-z^2}{2}\times\frac{2z}{bz^2+2az+b}\,\frac{dz}{iz}$$

$= R.P.\ \frac{1}{i}\int_C \frac{1-z^2}{bz^2+2az+b}\,dz = R.P\ \frac{1}{bi}\int_C \frac{1-z^2}{z^2+\frac{2a}{b}z+1}\,dz$



Consider, $z^2+\frac{2a}{b}z+1=0$

$$\therefore z = \frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b}\pm\frac{2}{b}\sqrt{a^2-b^2}}{2} = -\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b} = \frac{1}{b}\left[-a\pm\sqrt{a^2-b^2}\right]$$

Let $\alpha = \frac{1}{b}\left(-a+\sqrt{a^2-b^2}\right)$ & $\beta = \frac{1}{b}\left(-a-\sqrt{a^2-b^2}\right)$

Given: $a>b>0$. Let $b=1,\ a=2$

$\therefore \alpha = -2+\sqrt{4-1} = -2+\sqrt{3} = -2+1.732 = -0.268$ is a simple pole which lies inside c.

$\beta = -2-\sqrt{4-1} = -2-\sqrt{3} = -2-1.732 = -3.732$ is a simple pole which lies outside c.

$\underset{z=\alpha}{\text{Res}}\, f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{1-z^2}{(z-\alpha)(z-\beta)}$ Here $f(z) = \frac{1-z^2}{z^2+\frac{2a}{b}z+1} = \frac{1-z^2}{(z-\alpha)(z-\beta)}$

$$= \lim_{z\to\alpha}\frac{1-z^2}{z-\beta} = \frac{1-\alpha^2}{\alpha-\beta}$$

$$= \frac{1-\frac{1}{b^2}\left(-a+\sqrt{a^2-b^2}\right)^2}{\frac{1}{b}\left(-a+\sqrt{a^2-b^2}\right)-\frac{1}{b}\left(-a-\sqrt{a^2-b^2}\right)} = \frac{1-\frac{1}{b^2}\left(a^2+a^2-b^2-2a\sqrt{a^2-b^2}\right)}{\frac{1}{b}\left(-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}\right)}$$

$$= \frac{1-\frac{1}{b^2}\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}{\frac{1}{b}2\sqrt{a^2-b^2}} = \frac{b^2-2a^2+b^2+2a\sqrt{a^2-b^2}}{b^2\times\frac{1}{b}\times 2\sqrt{a^2-b^2}}$$

$$= \frac{2b^2-2a^2+2a\sqrt{a^2-b^2}}{2b\sqrt{a^2-b^2}} = \frac{b^2-a^2+a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = R.P.\ \frac{1}{i}\times 2\pi i\times \text{Sum of the residues}$$

$$= R.P\ 2\pi\times\frac{b^2-a^2+a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}} = \frac{2\pi}{b}\ \frac{-(a^2-b^2)+a\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{b}\left[\frac{-(a^2-b^2)}{\sqrt{a^2-b^2}}+\frac{a\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right] = \frac{2\pi}{b}\left[-\sqrt{a^2-b^2}+a\right]$$

$$= \frac{2\pi}{b}\left[a-\sqrt{a^2-b^2}\right]$$

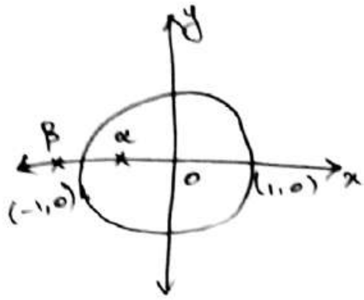
(31) Evaluate $\int_0^{2\pi}\frac{d\theta}{13+12\cos\theta}$ using contour integration.

Sol: Let $z=e^{i\theta}$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+12\cos\theta} = \int_c \frac{dz/iz}{13+12\,\frac{1}{2}\left(\frac{z^2+1}{z}\right)} \quad \text{where } c \text{ is } |z|=1$$

$$= \frac{1}{i}\int_c \frac{dz/z}{13+6\left(\frac{z^2+1}{z}\right)} = \frac{1}{i}\int_c \frac{z}{13z+6z^2+6}\,\frac{dz}{z}$$



$$= \frac{1}{i}\int_c \frac{dz}{6z^2+13z+6} = \frac{1}{6i}\int_c \frac{dz}{z^2+\frac{13}{6}z+1}$$

Consider, $z^2+\frac{13}{6}z+1=0$

$$\therefore z = \frac{-\frac{13}{6} \pm \sqrt{\frac{169}{36}-4}}{2} = \frac{-\frac{13}{6} \pm \sqrt{\frac{169-144}{36}}}{2} = \frac{-\frac{13}{6} \pm \sqrt{\frac{25}{36}}}{2} = \frac{-\frac{13}{6} \pm \frac{5}{6}}{2}$$

$$= \frac{-13 \pm 5}{12}$$

Let $\alpha = \frac{-13+5}{12} = \frac{-8}{12} = \frac{-2}{3}$ is a simple pole which lies inside c.

$\beta = \frac{-13-5}{12} = \frac{-18}{12} = \frac{-6}{4} = \frac{-3}{2} = -1.5$ is a simple pole which lies outside c.

$\therefore \operatorname{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} \frac{(z-\alpha)}{(z-\alpha)(z-\beta)}$. Here $f(z) = \frac{1}{z^2+\frac{13}{6}z+1} = \frac{1}{(z-\alpha)(z-\beta)}$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} = \frac{1}{\alpha-\beta} = \frac{1}{-2/3+3/2} = \frac{1}{\frac{-4+9}{6}} = \frac{6}{5}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+12\cos\theta} = \frac{1}{6i}\times 2\pi i \times \text{Sum of the residues} = \frac{1}{6i}\times 2\pi i\times\frac{6}{5} = \frac{2\pi}{5}$$

(32) Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ using contour integration.

Sol. Let $z=e^{i\theta}$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$; $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{1}{2i}\left(\frac{z^2-1}{z}\right)$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_c \frac{dz/iz}{13+5\frac{1}{2i}\left(\frac{z^2-1}{z}\right)} = \int_c \frac{2iz}{26iz+5z^2-5}\,\frac{dz}{iz} \quad \text{where } c \text{ is } |z|=1.$$

$$= 2\int_c \frac{dz}{5\left(z^2+\frac{i26}{5}z-1\right)} = \frac{2}{5}\int_c \frac{dz}{z^2+\frac{26i}{5}z-1}$$

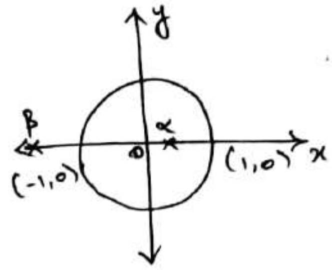
Consider, $z^2+\frac{26i}{5}z-1=0$

$$z = \frac{-\frac{26}{5} \pm \sqrt{\frac{676}{25}-4(-1)}}{2} = \frac{-\frac{26}{5} \pm \sqrt{\frac{776}{25}}}{2} = \frac{-\frac{26}{5} \pm \frac{2}{5}\sqrt{194}}{2} = \frac{-13}{5} \pm \frac{\sqrt{194}}{5}$$

Let $\alpha = \frac{1}{5}(-13+\sqrt{194})$ & $\beta = \frac{1}{5}(-13-\sqrt{194})$

$\alpha = 0.19$ is a simple pole which lies inside c.

$\beta = -5.39$ is a simple pole which lies outside c.



$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

Here $f(z) = \dfrac{1}{z^2+\frac{26}{5}iz-1} = \dfrac{1}{(z-\alpha)(z-\beta)}$

$$= \frac{1}{\alpha-\beta} = \frac{1}{\frac{1}{5}(-13+\sqrt{194}) - \frac{1}{5}(-13-\sqrt{194})}$$

$$= \frac{5}{-13+\sqrt{194}+13+\sqrt{194}} = \frac{5}{2\sqrt{194}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{2}{5}\times 2\pi i \times \text{Sum of the residues} = \frac{2}{5}\times 2\pi i \times \frac{5}{2\sqrt{194}} = \frac{2\pi i}{\sqrt{194}}$$

(33) Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\,d\theta$ using contour integration.

Sol: Let $z = e^{i\theta}$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

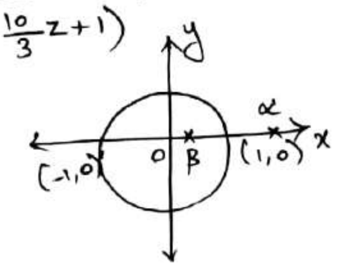
$\cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$; $\sin\theta = \frac{1}{2i}\left(\frac{z^2-1}{z}\right)$

$\sin^2\theta = \frac{1-\cos 2\theta}{2} = R.P\left(\frac{1-e^{i2\theta}}{2}\right) = R.P\left(\frac{1-z^2}{2}\right)$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}\,d\theta = R.P. \int_C \frac{\left(\frac{1-z^2}{2}\right)}{5-3\,\frac{1}{2}\left(\frac{z^2+1}{z}\right)}\frac{dz}{iz}$$

$$= R.P\int_C \frac{1-z^2}{2}\,\frac{2z}{10z-3z^2-3}\,\frac{dz}{iz} = R.P\,\frac{1}{i}\int_C \frac{1-z^2}{-3\left(z^2-\frac{10}{3}z+1\right)}dz$$

$$= R.P.\,\frac{1}{-3i}\int_C \frac{1-z^2}{z^2-\frac{10}{3}z+1}dz$$



Consider, $z^2 - \frac{10}{3}z + 1 = 0$

$$z = \frac{\frac{10}{3} \pm \sqrt{\frac{100}{9}-4}}{2} = \frac{\frac{10}{3}\pm\sqrt{\frac{100-36}{9}}}{2} = \frac{\frac{10}{3}\pm\sqrt{\frac{64}{9}}}{2} = \frac{\frac{10}{3}\pm\frac{8}{3}}{2} = \frac{5}{3}\pm\frac{4}{3}$$

Let $\alpha = \frac{1}{3}(5+4) = \frac{9}{3} = 3$ is a simple pole which lies outside c.

$\beta = \frac{1}{3}(5-4) = \frac{1}{3}$ is a simple pole which lies inside c.

$$\operatorname*{Res}_{z=\beta} f(z) = \lim_{z\to\beta}(z-\beta)\frac{1-z^2}{(z-\alpha)(z-\beta)} = \frac{1-\beta^2}{\beta-\alpha} = \frac{1-(1/3)^2}{1/3-3} = \frac{1-1/9}{-8/3} = \frac{8}{9}\times\frac{3}{-8} = \frac{-3}{9} = \frac{-1}{3}$$

$$\therefore \int_0^{2\pi}\frac{\sin^2\theta}{5-3\cos\theta} = R.P.\,\frac{1}{-3i}\times 2\pi i\times\frac{-1}{3} = \frac{2\pi}{9}$$

(34) Evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)}$ using contour integration.

Sol: WKT $2\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)} = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)}$

$\Rightarrow \int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)} = \frac{1}{2}\int_c \frac{z^2 dz}{(z^2+9)(z^2+4)}$ where c is the upper half of the semi-circle with diameter $(-R, R)$.

Here $z^2+9=0$, $z^2+4=0 \Rightarrow z^2=-4 \Rightarrow z=\sqrt{-4}=\pm 2i$ & $f(z) = \frac{z^2}{(z^2+9)(z^2+4)}$

$z^2=-9 \Rightarrow z=\sqrt{-9}=\pm 3i$

$\therefore z = \pm 2i, \pm 3i$

$z=2i$ is a simple pole lies inside c. $z=3i$ is a simple pole lies inside c.

$z=-2i$ is a simple pole lies outside c. $z=-3i$ is a simple pole lies outside c.

$$\underset{z=2i}{\text{Res}}\, f(z) = \lim_{z\to 2i} (z-2i)\frac{z^2}{(z-2i)(z+2i)(z+3i)(z-3i)} = \frac{(2i)^2}{(4i)(5i)(-i)} = \frac{-4}{(-20)(-i)}$$

$$= \frac{1}{-5i} = \frac{-1}{5i}$$

$$\underset{z=3i}{\text{Res}}\, f(z) = \lim_{z\to 3i} (z-3i)\frac{z^2}{(z-2i)(z+2i)(z+3i)(z-3i)} = \frac{(3i)^2}{(i)(5i)(6i)} = \frac{-9}{(-5)(6i)}$$

$$= \frac{-3}{-10i} = \frac{3}{10i}$$

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)} = \frac{1}{2}\times 2\pi i \times \text{Sum of the residues} = \frac{1}{2}\times 2\pi i \times\left(\frac{-1}{5i}+\frac{3}{10i}\right)$$

$$= \pi i \times\left(\frac{-2+3}{10i}\right) = \pi i \times \frac{1}{10i} = \frac{\pi}{10}$$

(35) Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$, a & b are +ve using contour integration.

Sol: WKT $2\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$

$\Rightarrow \int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_c \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)}$ where c is the upper half of the semicircle with diameter $(-R, R)$.

Here $z^2+a^2=0$ $\qquad z^2+b^2=0$ $\qquad$ & $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

$z^2=-a^2 \Rightarrow z=\sqrt{-a^2}=\pm ia$ $\qquad z^2=-b^2 \Rightarrow z=\sqrt{-b^2}=\pm bi$

$z=ai$ is a simple pole lies inside c. $z=bi$ is a simple pole lies inside c.

$z=-ai$ is a simple pole lies outside c. $z=-bi$ is a simple pole lies outside c.

$$\underset{z=ai}{\text{Res}}\, f(z) = \lim_{z\to ai} (z-ai)\frac{z^2}{(z-ai)(z+ai)(z-bi)(z+bi)} = \frac{(ai)^2}{(2ai)(ai-bi)(ai+bi)}$$

$$= \frac{-a^2}{(2ai)((ai)^2-(bi)^2)} = \frac{-a^2}{(2ai)(-a^2+b^2)} = \frac{a^2}{2ai(a^2-b^2)}$$

$$\underset{z=bi}{\text{Res}}\, f(z) = \lim_{z\to bi} (z-bi)\frac{z^2}{(z-ai)(z+ai)(z-bi)(z+bi)} = \frac{(bi)^2}{(bi-ai)(bi+ai)(2bi)}$$

$$= \frac{-b^2}{(2bi)(-b^2+a^2)} = \frac{-b^2}{(2bi)(a^2-b^2)}$$

$$\therefore \int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\times 2\pi i \times \text{Sum of the residues}$$

$$= \frac{1}{2}\times 2\pi i \times \left(\frac{a^2}{2ai(a^2-b^2)} - \frac{b^2}{2bi(a^2-b^2)}\right) = \frac{\pi i}{2i(a^2-b^2)}\left(\frac{a^2}{a}-\frac{b^2}{b}\right)$$

$$= \frac{\pi}{2(a^2-b^2)}(a-b) = \frac{\pi(a-b)}{2(a+b)(a-b)} = \frac{\pi}{2(a+b)}$$

---

(36) Show that $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{2}$.

Sol: Given $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \int_c \frac{z^2-z+2}{z^4+10z^2+9}\,dz$ where c is the upper half of the semicircle with diameter $(-R,R)$.

Consider, $z^4+10z^2+9=0$

$(z^2)^2+10z^2+9=0$

$$\begin{array}{c|c} \times & + \\ 9 & 10 \\ \hline 1 & 9 \\ z^2+1 & z^2+9 \end{array}$$

$\therefore (z^2+1)(z^2+9)=0$

$z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$ & $z^2+9=0 \Rightarrow z^2=-9 \Rightarrow z=\sqrt{-9}=\pm 3i$

$\therefore z=\pm i, \pm 3i$ Here $f(z)=\frac{z^2-z+2}{z^4+10z^2+9}$

$z=i$ is a simple pole lies inside c. $z=-i$ is a simple pole lies outside c.
$z=3i$ is a simple pole lies inside c. $z=-3i$ is a simple pole lies outside c.

$$\underset{z=i}{\text{Res}}\, f(z) = \lim_{z\to i}(z-i)\frac{z^2-z+2}{(z-i)(z+i)(z^2+9)} = \frac{i^2-i+2}{2i(i^2+9)} = \frac{-1-i+2}{16i} = \frac{1-i}{16i}$$

$$\underset{z=3i}{\text{Res}}\, f(z) = \lim_{z\to 3i}(z-3i)\frac{z^2-z+2}{(z^2+1)(z-3i)(z+3i)} = \frac{(3i)^2-3i+2}{((3i)^2+1)6i} = \frac{-9-3i+2}{-8\times 6i} = \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = 2\pi i\times\text{Sum of the residues} = 2\pi i\times\left(\frac{1-i}{16i}+\frac{7+3i}{48i}\right)$$

$$= 2\pi i\times\left(\frac{3-3i+7+3i}{48i}\right) = 2\pi i\times\frac{10}{48i} = \frac{10\pi}{24} = \frac{5\pi}{12}$$

---

(37) Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^2}$ using contour integration.

Sol: Given $2\int_0^\infty \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} \Rightarrow \int_0^\infty\frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = \frac{1}{2}\int_c\frac{dz}{(1+z^2)^2}$

where c is the upper half of the semicircle with diameter $(-R,R)$

Consider, $1+z^2=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$

$\therefore \int_c \frac{dz}{(1+z^2)^2} = \int_c \frac{dz}{(z+i)^2(z-i)^2}$. Here $f(z)=\frac{1}{(z+i)^2(z-i)^2}$

$z=i$ is a pole of order 2 lies inside c.

$z=-i$ is a pole of order 2 lies outside c.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{1}{(z+i)^2(z-i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[(z+i)^{-2}\right]$$

$$= \lim_{z\to i}(-2)(z+i)^{-3} = -2(2i)^{-3} = \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

$$\therefore \int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{1}{2}\times 2\pi i \times \text{Sum of the residues} = \pi i \times \frac{1}{4i} = \frac{\pi}{4}$$

(38) Evaluate $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx$ using contour integration.

Sol: Given $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx = \int_c \frac{z^2}{(z^2+1)^2}dz$ where c is the upper half of the semi-circle with diameter $(-R,R)$.

Consider, $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i$

$\therefore \int_c \frac{z^2}{(z^2+1)^2}dz = \int_c \frac{z^2}{(z+i)^2(z-i)^2}dz$. Here $f(z)=\frac{z^2}{(z+i)^2(z-i)^2}$

$z=i$ is a pole of order 2 lies inside c.

$z=-i$ is a pole of order 2 lies outside c.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{z^2}{(z+i)^2(z-i)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right]$$

$$= \lim_{z\to i}\left[\frac{(z+i)^2\cdot 2z - z^2\cdot 2(z+i)}{(z+i)^4}\right] = \lim_{z\to i}\left[\frac{(z+i)2z-2z^2}{(z+i)^3}\right]$$

$$= \frac{2i\cdot 2i - 2(i)^2}{(2i)^3} = \frac{-4+2}{-8i} = \frac{-2}{-8i} = \frac{1}{4i}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx = 2\pi i\times\text{Sum of the residues} = 2\pi i\times\frac{1}{4i} = \frac{\pi}{2}$$

(39) Evaluate $\int_0^\infty \frac{dx}{x^4+a^4}$ using contour integration.

Sol: Given $2\int_0^\infty \frac{dx}{x^4+a^4} = \int_{-\infty}^{\infty}\frac{dx}{x^4+a^4} \Rightarrow \int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4} = \frac{1}{2}\int_c \frac{dz}{z^4+a^4}$ where c is the upper half of the semi-circle with diameter $(-R,R)$

Consider, $z^4+a^4=0 \Rightarrow z^4=-a^4 \Rightarrow z=\sqrt[4]{-a^4}=a(-1)^{1/4}=a(\cos\pi+i\sin\pi)^{1/4}$

$\therefore z=a\left[\cos(2n+1)\pi+i\sin(2n+1)\pi\right]^{1/4}$ where $n=0,1,2,3$.

$= a\left(e^{i(2n+1)\pi}\right)^{1/4} = ae^{i(2n+1)\pi/4},\ n=0,1,2,3 = ae^{\frac{i\pi}{4}},\ ae^{\frac{i3\pi}{4}},\ ae^{\frac{i5\pi}{4}},\ ae^{\frac{i7\pi}{4}}$

$z = ae^{\frac{i\pi}{4}} = a(\cos\pi/4 + i\sin\pi/4) = a\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = a\left(\frac{1+i}{\sqrt{2}}\right)$ is a simple pole lies inside c.

$z = ae^{\frac{i3\pi}{4}} = a\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = a\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = a\left(\frac{-1+i}{\sqrt{2}}\right)$ is a simple pole lies inside c.

$z = ae^{\frac{i5\pi}{4}} = a\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = a\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = a\left(\frac{-1-i}{\sqrt{2}}\right)$ is a simple pole lies outside c.

$z = ae^{\frac{i7\pi}{4}} = a\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = a\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = a\left(\frac{1-i}{\sqrt{2}}\right)$ is a simple pole lies outside c.

$$\operatorname*{Res}_{z=ae^{i\pi/4}} f(z) = \lim_{z\to ae^{i\pi/4}} (z-ae^{i\pi/4}) f(z) = \lim_{z\to ae^{i\pi/4}} (z-ae^{i\pi/4}) \frac{1}{z^4+a^4} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to ae^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4(ae^{i\pi/4})^3} = \frac{1}{4a^3e^{i3\pi/4}}$$

$$\operatorname*{Res}_{z=ae^{i3\pi/4}} f(z) = \lim_{z\to ae^{\frac{i3\pi}{4}}} (z-ae^{i3\pi/4}) f(z) = \lim_{z\to ae^{\frac{i3\pi}{4}}} (z-ae^{\frac{i3\pi}{4}}) \frac{1}{z^4+a^4} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to ae^{\frac{i3\pi}{4}}} \frac{1}{4z^3} = \frac{1}{4(ae^{i3\pi/4})^3} = \frac{1}{4a^3e^{i9\pi/4}}$$

$$\therefore \int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2} \times 2\pi i \times \text{Sum of the residues}$$

$$= \pi i \times \left(\frac{1}{4a^3e^{i3\pi/4}} + \frac{1}{4a^3e^{i9\pi/4}}\right) = \frac{\pi i}{4a^3}\left[e^{-i3\pi/4} + e^{-i9\pi/4}\right]$$

$$= \frac{\pi i}{4a^3}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{\pi i}{4a^3}\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

$$= \frac{\pi i}{4a^3}\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \frac{\pi i}{4a^3} \times -2i\frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}a^3}$$

---

(40) Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^3}$, $a>0$ using contour integration.

Sol: WKT $2\int_0^\infty \frac{dx}{(x^2+a^2)^3} = \int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^3} \Rightarrow \int_0^\infty \frac{dx}{(x^2+a^2)^3} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)^3} = \frac{1}{2}\int_C \frac{dz}{(z^2+a^2)^3}$

where c is the upper half of the semi-circle with diameter $(-R, R)$.

Consider, $z^2+a^2=0 \Rightarrow z^2=-a^2 \Rightarrow z=\sqrt{-a^2} = \pm ai$

$z=ai$ is a pole of order 3 lies inside c. $(a=ai, m=3)$

$z=-ai$ is a pole of order 3 lies outside c.

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$\operatorname*{Res}_{z=ai} f(z) = \frac{1}{2!} \lim_{z\to ai} \frac{d^2}{dz^2}\left[(z-ai)^3 \frac{1}{(z-ai)^3(z+ai)^3}\right] = \frac{1}{2}\lim_{z\to ai} \frac{d^2}{dz^2}\left(\frac{1}{(z+ai)^3}\right)$$

$$= \frac{1}{2}\lim_{z\to ai}\frac{d^2}{dz^2}\left[(z+ai)^{-3}\right] = \frac{1}{2}\lim_{z\to ai}\frac{d}{dz}\left[-3(z+ai)^{-4}\right]$$

$$= \frac{1}{2}\lim_{z\to ai}\left[(-3)(-4)(z+ai)^{-5}\right] = \frac{1}{2}\times 12(2ai)^{-5} = \frac{6}{(2ai)^5} = \frac{6}{32a^5 i}$$

$$\therefore \int_0^\infty \frac{dx}{(x^2+a^2)^3} = \frac{1}{2}\times 2\pi i \times \text{Sum of the residues} = \pi i \times \frac{6}{32a^5 i} = \frac{3\pi}{16a^5}$$

---

(41) Evaluate $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$ using contour integration.

Sol: WKT $2\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \int_{-\infty}^\infty \frac{\cos mx}{x^2+a^2}\,dx \Rightarrow \int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{1}{2}\int_C \frac{\cos mz}{z^2+a^2}\,dz$

where c is the upper half of the semi-circle with diameter $(-R, R)$.

Consider $z^2+a^2=0 \Rightarrow z^2=-a^2 \Rightarrow z=\sqrt{-a^2} = \pm ai$

$z=ai$ is a simple pole lies inside c. $z=-ai$ is a simple pole lies outside c.

$$\therefore \int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \text{R.P. } \frac{1}{2}\int_C \frac{e^{imz}}{z^2+a^2}\,dz. \quad \text{Here } f(z) = \frac{e^{imz}}{z^2+a^2}$$

$$\operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai}(z-ai)\frac{e^{imz}}{(z-ai)(z+ai)} = \frac{e^{im(ai)}}{2ai} = \frac{e^{-am}}{2ai}$$

$$\therefore \int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \text{R.P. } \frac{1}{2}\times 2\pi i \times \text{Sum of the residues} = \text{R.P. } \frac{1}{2}\times 2\pi i \times \frac{e^{-am}}{2ai} = \frac{e^{-am}\pi}{2a}$$

---

(42) Evaluate $\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx$ where $a>0$, $m>0$ using contour integration.

Sol: WKT $2\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \int_{-\infty}^\infty \frac{x\sin mx}{x^2+a^2}\,dx$

$$\Rightarrow \int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \frac{1}{2}\int_C \frac{z\sin mz}{z^2+a^2}\,dz \text{ where } c \text{ is the}$$

upper half of the semi-circle with diameter $(-R, R)$.

Consider, $z^2+a^2=0 \Rightarrow z^2=-a^2 \Rightarrow z=\sqrt{-a^2} = \pm ai$

$z=ai$ is a simple pole lies inside c. $z=-ai$ is a simple pole lies outside c.

$$\therefore \int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \text{Im } \frac{1}{2}\int_C \frac{z e^{imz}}{z^2+a^2}\,dz. \quad \text{Here } f(z) = \frac{z e^{imz}}{z^2+a^2}$$

$$\operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai}(z-ai)\frac{z e^{imz}}{(z-ai)(z+ai)} = \frac{ai\, e^{im(ai)}}{2ai} = \frac{e^{-am}}{2}$$

$$\therefore \int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \text{Im } \frac{1}{2}\times 2\pi i \times \text{Sum of the residues} = \text{Im } \pi i \times \frac{e^{-am}}{2} = \frac{\pi e^{-am}}{2}$$

DIFFERENTIAL EQUATIONS

Complementary function: (C.F.)

Roots of Auxiliary equation	Complementary function
(1) Roots are real & distinct. m_1, m_2 $(m_1 \neq m_2)$	(1) $Ae^{m_1 x} + Be^{m_2 x}$
(2) Roots are real & equal. m_1, m_2 $(m_1 = m_2)$	(2) $Ae^{m_1 x} + xBe^{m_2 x}$
(3) Roots are complex. (ie) $\alpha \pm i\beta$	(3) $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Particular Integral: (P.I.)

RHS	P.I.
(1) e^{ax}	(1) $D = a$
(2) $\sin ax$ (or) $\cos ax$	(2) $D^2 = -a^2$
(3) x^n	(3) $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$
(4) $e^{ax} f(x)$	(4) $\frac{1}{f(D)} e^{ax} f(x) = e^{ax} \frac{1}{f(D+a)} f(x)$

---

(1) **Solve:** $(D^2 - 5D + 6)y = 0$

Sol: Auxiliary equation is

×	+
6	−5
−3	−2
m−3	m−2

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m = 2, 3$$

$\therefore$ C.F. is $y = Ae^{2x} + Be^{3x}$.

---

(2) **Solve:** $(D^2 + 6D + 9)y = 0$

Sol: Auxiliary equation is

×	+
9	6
3	3
m+3	m+3

$$m^2 + 6m + 9 = 0$$

$$(m+3)(m+3) = 0 \implies m = -3, -3$$

$\therefore$ C.F. is $y = Ae^{-3x} + xBe^{-3x}$.

③ **Solve:** $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$.

Sol: Given $(D^2 - 6D + 13)y = 0$

Auxiliary equation is $m^2 - 6m + 13 = 0$

$$\therefore m = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = \frac{6 \pm \sqrt{36-52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i \quad (\alpha = 3, \beta = 2)$$

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ a = 1, b = -6, c = 13 \end{array}}$$

$\therefore$ C.F. is $y = e^{3x}(A\cos 2x + B\sin 2x)$.

④ **Solve:** $(D^3 - 3D^2 + 3D - 1)y = 0$

Sol: Auxiliary equation is

$m^3 - 3m^2 + 3m - 1 = 0$

$$\begin{array}{r|rrrr} m=1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$\times$	$+$
1	-2
-1	-1
$m-1$	$m-1$

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0 \Rightarrow m = 1, 1$

$\therefore m = 1, 1, 1$.

$\therefore$ C.F. is $y = Ae^{x} + xBe^{x} + x^2Ce^{x}$.

⑤ **Solve:** $(D^3 + D^2 + 4D + 4)y = 0$.

Sol: Auxiliary equation is

$m^3 + m^2 + 4m + 4 = 0$

$$\begin{array}{r|rrrr} m=-1 & 1 & 1 & 4 & 4 \\ & & -1 & 0 & -4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \sqrt{-4} = \pm 2i$

$\therefore m = -1, \pm 2i$

$\therefore$ C.F. is $y = Ae^{-x} + B\cos 2x + C\sin 2x$.

⑥ **Solve:** $(D^4-2D^2+1)y=0$

Sol Auxiliary equation is

$$m^4-2m^2+1=0$$

$$(m^2)^2-2m^2+1=0$$

$$(m^2-1)(m^2-1)=0$$

1	−2
−1	−1
m^2-1	m^2-1

$$m^2-1=0 \Rightarrow m^2=1 \Rightarrow m=\pm\sqrt{1}=\pm 1$$

$$\therefore m=\pm1,\pm1 = 1,-1,1,-1$$

$\therefore$ C.F. is $y=Ae^{x}+xBe^{x}+Ce^{-x}+xDe^{-x}$.

(H.W) **Solve the following:**

① $(D^3-D^2-D-2)y=0$

② $(D^2+2D+1)y=0$

③ $(D^2+1)y=0$

④ $(D^3+1)y=0$

⑤ $(D-1)^2y=0$

(H.W) ④ $(D^3+1)y=0$

Auxiliary equation is

$$m^3+1=0$$

$$\begin{array}{r|rrrr} m=-1 & 1 & 0 & 0 & 1 \\ & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$$m^2-m+1=0$$

$$m=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2}$$

$$(\alpha=1/2,\ \beta=\sqrt{3}/2)$$

$$\therefore y=Ae^{-x}+e^{\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x+C\sin\frac{\sqrt{3}}{2}x\right)$$

⑦ **Solve:** $(D^2+4D+5)y=e^{x}+x^3+\cos 2x+1$.

Sol: Auxiliary equation is

$$m^2+4m+5=0.$$

$$\boxed{a=1,\ b=4,\ c=5 \quad m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$m=\frac{-4\pm\sqrt{16-4(5)}}{2}=\frac{-4\pm\sqrt{16-20}}{2}$$

$$=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=-2\pm i \quad (\alpha=-2,\ \beta=1)$$

$\therefore$ C.F. is $e^{-2x}(A\cos x+B\sin x)$.

$$(P.I)_1=\frac{1}{D^2+4D+5}e^{x} \qquad \boxed{D=a=1}$$

$$=\frac{1}{1+4+5}e^{x}=\frac{1}{10}e^{x}$$

$$(P.I)_2=\frac{1}{D^2+4D+5}x^3=\frac{1}{5\left(\frac{D^2+4D}{5}+1\right)}x^3$$

$$\boxed{(1+x)^{-1}=1-x+x^2-x^3+\cdots}$$

$$=\frac{1}{5}\left[1+\frac{D^2+4D}{5}\right]^{-1}x^3$$

$$=\frac{1}{5}\left[1-\frac{D^2+4D}{5}+\left(\frac{D^2+4D}{5}\right)^2-\left(\frac{D^2+4D}{5}\right)^3+\cdots\right]x^3$$

$$\boxed{\begin{aligned} D&=3x^2\\ D^2&=6x\\ D^3&=6\end{aligned}}$$

$$=\frac{1}{5}\left[1-\frac{D^2+4D}{5}+\frac{1}{25}\left(D^4+16D^2+8D^3\right)-\frac{1}{125}\left(D^6+12D^5+48D^4+64D^3\right)+\cdots\right]x^3$$

$$\boxed{(a+b)^3=a^3+3a^2b+3ab^2+b^3}$$

$$= \frac{1}{5}\left[x^3 - \frac{6x+4(3x^2)}{5} + \frac{1}{25}\left(16(6x)+8(6)\right) - \frac{1}{125}\left(64(6)\right)\right]$$

$$= \frac{1}{5}\left[x^3 - \frac{6x}{5} - \frac{12x^2}{5} + \frac{96x}{25} + \frac{48}{25} - \frac{384}{125}\right]$$

$$= \frac{1}{5}\left[x^3 - \frac{12x^2}{5} + x\left(\frac{96}{25} - \frac{6}{5}\right) + \frac{48}{25} - \frac{384}{125}\right]$$

$$= \frac{1}{5}\left[x^3 - \frac{12x^2}{5} + \frac{66}{25}x - \frac{144}{125}\right]$$

$$(P.I)_3 = \frac{1}{D^2+4D+5}\cos 2x \qquad \boxed{D^2 = -a^2 = -4}$$

$$= \frac{1}{-4+4D+5}\cos 2x = \frac{1}{4D+1}\cos 2x$$

$$= \frac{4D-1}{(4D+1)(4D-1)}\cos 2x = \frac{(4D-1)}{(4D)^2-1^2}\cos 2x = \frac{(4D-1)}{16D^2-1}\cos 2x$$

$$= \frac{(4D-1)}{16(-4)-1}\cos 2x = \frac{-1}{65}\left[4(-\sin 2x \cdot 2) - \cos 2x\right]$$

$$= \frac{1}{65}(8\sin 2x + \cos 2x)$$

$$(P.I)_4 = \frac{1}{D^2+4D+5}(1) = \frac{1}{D^2+4D+5}e^{0x} \qquad \boxed{D = a = 0}$$

$$= \frac{1}{5}e^{0x} = \frac{1}{5}$$

Hence the general solution is

$y = C.F + P.I$

$$y = e^{-2x}(A\cos x + B\sin x) + \frac{e^x}{10} + \frac{1}{5}\left[x^3 - \frac{12x^2}{5} + \frac{66}{25}x - \frac{144}{125}\right]$$

$$+ \frac{1}{65}(8\sin 2x + \cos 2x) + \frac{1}{5}.$$

---

(8) Solve: $(D-2)^2 y = e^{2x}$.

Sol: Auxiliary equation is $(m-2)^2 = 0$

$\Rightarrow (m-2)(m-2) = 0$

$\Rightarrow m = 2, 2$

$\therefore$ C.F. is $Ae^{2x} + xBe^{2x}$

$$P.I = \frac{1}{(D-2)^2} e^{2x} = \frac{x}{2(D-2)} e^{2x} = \frac{x^2}{2} e^{2x}$$

$$\therefore y = C.F + P.I = Ae^{2x} + xBe^{2x} + \frac{x^2}{2} e^{2x}.$$

(9) Find the particular integral of $(D^3+1)y = \cos(2x-1)$.

Sol: $$P.I = \frac{1}{D^3+1} \cos(\underset{a}{2x}-1)$$

$$\boxed{D^2 = -a^2 = -2^2 = -4}$$

$$= \frac{1}{-4 \cdot D+1} \cos(2x-1) = \frac{1}{1-4D} \cos(2x-1)$$

$$= \frac{1+4D}{(1-4D)(1+4D)} \cos(2x-1) = \frac{1+4D}{1^2-(4D)^2} \cos(2x-1)$$

$$= \frac{1+4D}{1-16D^2} \cos(2x-1) = \frac{1+4D}{1-16(-4)} \cos(2x-1) = \frac{1+4D}{65} \cos(2x-1)$$

$$= \frac{\cos(2x-1) + 4[-\sin(2x-1) \cdot 2]}{65} = \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

AU (10) Find the particular integral of $(D-a)^2 y = e^{ax} \sin x$.

Sol: $$P.I = \frac{1}{(D-a)^2} e^{ax} \sin x$$

$$\boxed{D \to D+a}$$

$$\boxed{\frac{1}{D} \to \text{Integrate with respect to } x}$$

$$= \frac{e^{ax}}{(D+a-a)^2} \sin x = \frac{e^{ax}}{D^2} \sin x = \frac{e^{ax}}{D}(-\cos x)$$

$$= e^{ax}(-\sin x) = -e^{ax} \sin x$$

(11) Solve: $(D^2-2D+2)y = e^x x^2 + 5 + e^{-2x}$.

Sol: Auxiliary equation is $m^2 - 2m + 2 = 0$

$$\boxed{a=1, b=-2, c=2 \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i \quad (\alpha = 1, \beta = 1)$$

$\therefore$ C.F. is $e^x(A\cos x + B\sin x)$.

$$(P.I)_1 = \frac{1}{D^2-2D+2} e^x x^2$$

$$\boxed{D \to D+1}$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2 = e^x \frac{1}{D^2+1+2D-2D-2+2} x^2 = e^x \frac{1}{D^2+1} x^2$$

$$= e^{x}(1+D^2)^{-1}x^2 = e^{x}\left[1-D^2+(D^2)^2-\cdots\right]x^2 \quad (\because (1+x)^{-1} = 1-x+x^2-\cdots)$$

$$= e^{x}\left[1-D^2+D^4-\cdots\right]x^2$$

$$\boxed{\begin{matrix} D = 2x \\ D^2 = 2 \end{matrix}}$$

$$= e^{x}\left[x^2-2\right]$$

$$(P.I)_2 = \frac{1}{D^2-2D+2}\,5 = 5\,\frac{1}{D^2-2D+2}\,e^{0x} \qquad \boxed{D = a = 0}$$

$$= \frac{5}{2}e^{0x} = \frac{5}{2}$$

$$(P.I)_3 = \frac{1}{D^2-2D+2}\,e^{-2x} = \frac{1}{(-2)^2-2(-2)+2}\,e^{-2x} \qquad \boxed{D = a = -2}$$

$$= \frac{1}{4+4+2}\,e^{-2x} = \frac{1}{10}e^{-2x}$$

$$\therefore y = C.F + (P.I)_1 + (P.I)_2 + (P.I)_3$$

$$\therefore y = e^{x}(A\cos x + B\sin x) + e^{x}(x^2-2) + \frac{5}{2} + \frac{e^{-2x}}{10}.$$

A.U (1)

1. Find the P.I of $(D-1)^2 y = e^{x}\sin x$.
2. Solve: $(D^4-2D^3+D^2)y = x^3$.
3. Find the P.I of $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$.

Cauchy-Euler's Type:

(AU)

(12) Solve the equation $x^2y'' - xy' + y = 0$.

Sol: Given $(x^2D^2 - xD + 1)y = 0$ — (1)

Put $x = e^{z}$

$\log x = \log e^{z} = z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$$\therefore (1) \Rightarrow [D'(D'-1) - D' + 1]y = 0 \Rightarrow (D'^2 - D' - D' + 1)y = 0$$

$$\Rightarrow (D'^2 - 2D' + 1)y = 0$$

$\times$	$+$
1	-2
-1	-1
$m-1$	$m-1$

Auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$\therefore$ C.F. is $y = Ae^{z} + zBe^{z}$

$$y = Ax + \log x\,Bx = Ax + Bx\log x$$

(AU) 13) Convert $x^2y''-2xy'+2y=0$ into a linear differential equation with constant coefficients.

Sol: Given $(x^2D^2-2xD+2)y=0$ — ①

Put $x=e^z$ | $xD=D'$

$\log x = z$ | $x^2D^2=D'(D'-1)$

$\therefore ① \Rightarrow [D'(D'-1)-2D'+2]y=0$

$\Rightarrow (D'^2-D'-2D'+2)y=0 \Rightarrow (D'^2-3D'+2)y=0.$

(AU) ① Convert $xy''+y'=0$ into a linear differential equation with constant coefficients.

(AU) ② Transform the equation $xy''+y'+1=0$ into a linear equation with constant coefficients. [Hint: $xy''+y'=-1$]

$\boxed{\ln x = \log x}$

(AU) 14) Solve: $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=\left(\dfrac{\ln x}{x}\right)^2.$

Sol: Given $(x^2D^2-xD+1)y=\left(\dfrac{\log x}{x}\right)^2$ — ①

Put $x=e^z$ | $xD=D'$

$\log x = z$ | $x^2D^2=D'(D'-1)$

$\therefore ① \Rightarrow [D'(D'-1)-D'+1]y=\left(\dfrac{z}{e^z}\right)^2$

$(D'^2-D'-D'+1)y=\dfrac{z^2}{e^{2z}}=z^2e^{-2z}$

$(D'^2-2D'+1)y=e^{-2z}z^2$

Auxiliary equation is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1,1$

C.F. is $Ae^z+zBe^z = Ax+\log x\,Bx = Ax+Bx\log x$

$\boxed{D'\to D'-2}$

$$P.I = \frac{1}{D'^2-2D'+1}e^{-2z}z^2$$

$$=e^{-2z}\frac{1}{(D'-2)^2-2(D'-2)+1}z^2$$

$$=e^{-2z}\frac{1}{D'^2+4-4D'-2D'+4+1}z^2 = e^{-2z}\frac{1}{D'^2-6D'+9}z^2$$

$$=\frac{e^{-2z}}{9}\left[\frac{1}{\frac{D'^2-6D'}{9}+1}\right]z^2 = \frac{e^{-2z}}{9}\left[1+\frac{D'^2-6D'}{9}\right]^{-1}z^2$$

$$= \frac{e^{-2z}}{9}\left[1-\left(\frac{D'^2-6D'}{9}\right)+\left(\frac{D'^2-6D'}{9}\right)^2-\cdots\right]z^2 \quad \left(\because (1+x)^{-1}=1-x+x^2-\cdots\right)$$

$$= \frac{e^{-2z}}{9}\left[1-\left(\frac{D'^2-6D'}{9}\right)+\left(\frac{1}{81}\left(D'^4+36D'^2-12D'^3\right)\right)-\cdots\right]z^2 \quad \boxed{\begin{matrix} D'=2z \\ D'^2=2 \end{matrix}}$$

$$= \frac{e^{-2z}}{9}\left[z^2-\frac{1}{9}(2-12z)+\frac{1}{81}(36\times 2)\right] = \frac{e^{-2z}}{9}\left(z^2-\frac{2}{9}+\frac{12}{9}z+\frac{8}{9}\right)$$

$$= \frac{e^{-2z}}{9}\left(z^2+\frac{12}{9}z+\frac{6}{9}\right) = \frac{e^{-2z}}{81}\left(9z^2+12z+6\right) = \frac{e^{-2z}}{27}\left(3z^2+4z+2\right)$$

$$= \frac{e^{-2\log x}}{27}\left(3(\log x)^2+4\log x+2\right) = \frac{1}{27x^2}\left(3(\log x)^2+4\log x+2\right)$$

$$\therefore y = C.F + P.I = Ax + Bx\log x + \frac{1}{27x^2}\left(3(\log x)^2+4\log x+2\right)$$

(A) Legendre's Type:

$$\boxed{\begin{matrix} (ax+b)D = aD' \\ (ax+b)^2D^2 = a^2D'(D'-1) \end{matrix}}$$

(15) Solve: $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y = 3x+4.$

Sol. Given $\left[(x+2)^2D^2-(x+2)D+1\right]y = 3x+4$ — (1)

Put $x+2=e^z \Rightarrow x=e^z-2$, $\log(x+2)=z$ | $(x+2)D=D'$, $(x+2)^2D^2=D'(D'-1)$

$$\therefore (1) \Rightarrow \left[D'(D'-1)-D'+1\right]y = 3(e^z-2)+4 = 3e^z-6+4 = 3e^z-2$$

$$(D'^2-D'-D'+1)y = 3e^z-2 \Rightarrow (D'^2-2D'+1)\,y = 3e^z-2$$

Auxiliary equation is $m^2-2m+1=0$

$$(m-1)(m-1)=0 \Rightarrow m=1,1$$

$$\therefore \text{C.F. is } Ae^z+zBe^z = A(x+2)+\log(x+2)B(x+2)$$

$$= A(x+2)+B(x+2)\log(x+2)$$

$$(P.I)_1 = \frac{1}{(D'^2-2D'+1)}3e^z = 3\frac{1}{1-2+1}e^z \quad \boxed{D'=a=1}$$

$$= 3\frac{z}{2D'-2}e^z = \frac{3z^2}{2}e^z = \frac{3\left(\log(x+2)\right)^2}{2}(x+2)$$

$$(P.I)_2 = \frac{1}{D'^2-2D'+1}\left(-2e^{0z}\right) = \frac{1}{1}\left(-2e^{0z}\right) = -2 \quad \boxed{D'=a=0}$$

$$\therefore y = C.F + P.I$$

$$y = A(x+2)+B(x+2)\log(x+2)+\frac{3}{2}(x+2)\left(\log(x+2)\right)^2-2$$

(AU) (16) Solve: $[(x+1)^2D^2+(x+1)D+1]y=4\cos\log(x+1)$.

Sol: Put $x+1=e^z$ | $(x+1)D=D'$

$\log(x+1)=z$ | $(x+1)^2D^2=D'(D'-1)$

$[D'(D'-1)+D'+1]y=4\cos z \Rightarrow (D'^2-D'+D'+1)y=4\cos z$

$\Rightarrow (D'^2+1)y=4\cos z$

Auxiliary equation is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\sqrt{-1}=\pm i$

$\therefore m=\pm i \quad (\alpha=0,\beta=1)$

C.F is $e^{0z}(A\cos z+B\sin z)=A\cos z+B\sin z$

$=A\cos\log(x+1)+B\sin\log(x+1)$

$\boxed{D'^2=-a^2=-1}$

$$P.I=\frac{1}{D'^2+1}4\cos z=\frac{1}{-1+1}4\cos z$$

$$=4\frac{z}{2D'}\cos z=4z\frac{D'}{2D'^2}\cos z=4z\frac{D'}{2(-1)}\cos z$$

$$=-2zD'(\cos z)=-2z(-\sin z)=2z\sin z=2\log(x+1)\sin\log(x+1)$$

$\therefore y=C.F+P.I=A\cos\log(x+1)+B\sin\log(x+1)+2\log(x+1)\sin\log(x+1)$

(AU) (17) Solve: $\frac{dx}{dt}+\frac{dy}{dt}+3x=\sin t,\ \frac{dx}{dt}+y-x=\cos t$. — (*)

Sol: Given $Dx+Dy+3x=\sin t \Rightarrow (D+3)x+Dy=\sin t$ — (1)

$Dx+y-x=\cos t \Rightarrow (D-1)x+y=\cos t$ — (2)

(2)$\times D \Rightarrow D(D-1)x+\cancel{Dy}=D(\cos t)=-\sin t$

$(D+3)x+\cancel{Dy}=\sin t$ — (1)

$(-) \quad (-) \quad (-)$

$(D(D-1)-(D+3))x=-\sin t-\sin t$

$(D^2-D-D-3)x=-2\sin t \Rightarrow (D^2-2D-3)x=-2\sin t$

Auxiliary equation is $m^2-2m-3=0$

$(m+1)(m-3)=0$

$\Rightarrow m=-1,3$

-3	-2
+1	-3
m+1	m-3

$\therefore$ C.F. is $Ae^{-t}+Be^{3t}$

$\boxed{D^2=-a^2=-1}$

$$P.I=\frac{1}{D^2-2D-3}(-2\sin t)=-2\frac{1}{D^2-2D-3}\sin t$$

$$=-2\frac{1}{-1-2D-3}\sin t=-2\frac{1}{-2D-4}\sin t=\frac{-2}{-2}\frac{1}{D+2}\sin t$$

$$= \frac{D-2}{(D+2)(D-2)}\sin t = \frac{D-2}{D^2-2^2}\sin t = \frac{D-2}{-1-4}\sin t = \frac{-1}{5}(D-2)\sin t$$

$$= \frac{-1}{5}(\cos t - 2\sin t)$$

$\therefore x = C.F + P.I$

$$\boxed{x = Ae^{-t} + Be^{3t} - \frac{1}{5}(\cos t - 2\sin t)} \quad (3)$$

$$\frac{dx}{dt} = Ae^{-t}(-1) + Be^{3t}(3) - \frac{1}{5}(-\sin t - 2\cos t)$$

$$\frac{dx}{dt} = -Ae^{-t} + 3Be^{3t} + \frac{1}{5}(\sin t + 2\cos t) \quad (4)$$

Substituting (3) & (4) in (*),

$$-Ae^{-t} + 3Be^{3t} + \frac{1}{5}(\sin t + 2\cos t) + y - Ae^{-t} - Be^{3t} + \frac{1}{5}(\cos t - 2\sin t) = \cos t$$

$$y = Ae^{-t} - 3Be^{3t} - \frac{1}{5}(\sin t + 2\cos t) + Ae^{-t} + Be^{3t} - \frac{1}{5}(\cos t - 2\sin t) + \cos t$$

$$y = 2Ae^{-t} - 2Be^{3t} - \frac{1}{5}\sin t - \frac{2}{5}\cos t - \frac{1}{5}\cos t + \frac{2}{5}\sin t + \cos t$$

$$y = 2Ae^{-t} - 2Be^{3t} + \frac{1}{5}\sin t - \frac{3}{5}\cos t + \cos t$$

$$\boxed{y = 2Ae^{-t} - 2Be^{3t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t}$$

(AU) (18) Solve the simultaneous differential equation $Dx + y = \sin 2t$ & $-x + Dy = \cos 2t$.

Sol: Given $Dx + y = \sin 2t$ — (1)

$-x + Dy = \cos 2t$ — (2)

$$(2) \times D \Rightarrow -\cancel{Dx} + D^2 y = D(\cos 2t) = -2\sin 2t$$

$$\cancel{Dx} + y = \sin 2t$$

$$(D^2+1)y = -2\sin 2t + \sin 2t = -\sin 2t$$

$$(D^2+1)y = -\sin 2t$$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \sqrt{-1} = \pm i \ (\alpha = 0, \beta = 1)$

C.F. is $e^{0t}(A\cos t + B\sin t) = A\cos t + B\sin t$

$$\boxed{D^2 = -a^2 = -4}$$

$$P.I. = \frac{1}{D^2+1}(-\sin 2t) = -\frac{1}{-4+1}\sin 2t = \frac{1}{3}\sin 2t$$

$$\therefore y = C.F + P.I = A\cos t + B\sin t + \frac{1}{3}\sin 2t$$

$$Dy = A(-\sin t) + B\cos t + \frac{1}{3}(\cos 2t).2 = -A\sin t + B\cos t + \frac{2}{3}\cos 2t$$

$$\therefore (2) \Rightarrow x = -A\sin t + B\cos t + \frac{2}{3}\cos 2t - \cos 2t = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$$

$\therefore x = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$

$y = A\cos t + B\sin t + \frac{1}{3}\sin 2t$

(AU) (19) Solve: $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$, $\frac{dx}{dt} - 2x + \frac{dy}{dt} = \sin 2t$

Sol: Given $Dx - Dy + 2y = \cos 2t \Rightarrow Dx - (D-2)y = \cos 2t$ — (1)

$Dx - 2x + Dy = \sin 2t \Rightarrow (D-2)x + Dy = \sin 2t$ — (2)

$(1)\times D \Rightarrow D^2x - D(D-2)y = D(\cos 2t) = -2\sin 2t$

$(2)\times(D-2) \Rightarrow (D-2)^2x + D(D-2)y = (D-2)\sin 2t = 2\cos 2t - 2\sin 2t$

$[D^2 + (D-2)^2]x = -2\sin 2t + 2\cos 2t - 2\sin 2t$

$(D^2 + D^2 + 4 - 4D)x = -4\sin 2t + 2\cos 2t$

$(2D^2 - 4D + 4)x = -4\sin 2t + 2\cos 2t$

$(D^2 - 2D + 2)x = -2\sin 2t + \cos 2t$

Auxiliary equation is $m^2 - 2m + 2 = 0$

$$\boxed{a=1,\ b=-2,\ c=2 \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

$\therefore m = \frac{2 \pm \sqrt{4-4(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2} = 1 \pm i \quad (\alpha = 1, \beta = 1)$

$\therefore$ C.F. is $e^t(A\cos t + B\sin t)$

$$\boxed{D^2 = -a^2 = -4}$$

$(P.I)_1 = \frac{1}{D^2 - 2D + 2}\sin 2t = \frac{1}{-4 - 2D + 2}\sin 2t$

$= \frac{1}{-2-2D}\sin 2t = \frac{-1}{2}\frac{1}{1+D}\sin 2t = \frac{-1}{2}\frac{1-D}{(1+D)(1-D)}\sin 2t$

$= \frac{-1}{2}\frac{(1-D)}{1-D^2}\sin 2t = \frac{-1}{2}\frac{(1-D)}{1-(-4)}\sin 2t = \frac{-1}{10}(1-D)\sin 2t$

$= \frac{-1}{10}(\sin 2t - 2\cos 2t)$

$$\boxed{D^2 = -a^2 = -4}$$

$(P.I)_2 = \frac{1}{D^2 - 2D + 2}\cos 2t = \frac{1}{-4 - 2D + 2}\cos 2t$

$= \frac{1}{-2D-2}\cos 2t = \frac{-1}{2}\frac{1}{D+1}\cos 2t = \frac{-1}{2}\frac{D-1}{(D-1)(D+1)}\cos 2t$

$= \frac{-1}{2}\frac{(D-1)}{D^2-1}\cos 2t = \frac{-1}{2}\frac{(D-1)}{-4-1}\cos 2t = \frac{1}{10}(-2\sin 2t - \cos 2t)$

$\therefore x = C.F + P.I$

$x = e^t(A\cos t + B\sin t) + \frac{2}{10}(\sin 2t - 2\cos 2t) + \frac{1}{10}(-2\sin 2t - \cos 2t)$

$= e^t(A\cos t + B\sin t) + \frac{2}{10}\sin 2t - \frac{2}{5}\cos 2t - \frac{1}{5}\sin 2t - \frac{1}{10}\cos 2t$

$$\boxed{x = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t}$$

① $\times (D-2) \Rightarrow D(\cancel{D-2})x - (D-2)^2 y = (D-2)\cos 2t = -2\sin 2t - 2\cos 2t$

② $\times D \quad \Rightarrow D(D-2)x + D^2 y \quad = D(\sin 2t) = 2\cos 2t$

$\quad (-) \qquad (-) \qquad (-)$

$-[(D-2)^2 + D^2]y = -2\sin 2t - 2\cos 2t - 2\cos 2t$

$-[D^2 + 4 - 4D + D^2]y = -2\sin 2t - 4\cos 2t$

$(2D^2 - 4D + 4)y = 2\sin 2t + 4\cos 2t$

$(D^2 - 2D + 2)y = \sin 2t + 2\cos 2t$

$\therefore y = e^t(A\cos t + B\sin t) - \frac{1}{10}(\sin 2t - 2\cos 2t) + \frac{2}{10}(-2\sin 2t - \cos 2t)$

$= e^t(A\cos t + B\sin t) - \frac{1}{10}\sin 2t + \frac{1}{5}\cos 2t - \frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t$

$$\boxed{y = e^t(A\cos t + B\sin t) - \frac{1}{2}\sin 2t}$$

Method of variation of parameters:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X \quad \text{—①}$$

$C.F = Af_1 + Bf_2$ where f_1 & f_2 are functions of x & A & B are constants.

$P.I = Pf_1 + Qf_2$ where

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx \quad \& \quad Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx$$

$\therefore y = C.F + P.I = Af_1 + Bf_2 + Pf_1 + Qf_2$

Note: Wronskian of f_1, f_2 of equation ① is given by

$$w = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = f_1 f_2' - f_1' f_2$$

(AU) (20) Solve by the method of variation of parameters: $\frac{d^2y}{dx^2}+a^2y=\tan ax$.

Sol: Given $(D^2+a^2)y=\tan ax$. Here $X=\tan ax$

Auxiliary equation is $m^2+a^2=0$

$m^2=-a^2 \Rightarrow m=\sqrt{-a^2}=\pm ai \quad (\alpha=0,\ \beta=a)$

$\therefore$ C.F. is $A\cos ax+B\sin ax$

Here $f_1=\cos ax$, $f_2=\sin ax$

$f_1'=-a\sin ax$, $f_2'=a\cos ax$

$f_1f_2'-f_1'f_2=\cos ax(a\cos ax)-(-a\sin ax)\sin ax$

$=a\cos^2 ax+a\sin^2 ax=a(\cos^2 ax+\sin^2 ax)=a$

$$P=-\int\frac{f_2X}{f_1f_2'-f_1'f_2}dx=-\int\frac{\sin ax\tan ax}{a}dx$$

$$=\frac{-1}{a}\int\sin ax\,\frac{\sin ax}{\cos ax}dx=\frac{-1}{a}\int\frac{\sin^2 ax}{\cos ax}dx=\frac{-1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx$$

$$=\frac{-1}{a}\int(\sec ax-\cos ax)dx=\frac{-1}{a}\left[\frac{\log(\sec ax+\tan ax)}{a}-\frac{\sin ax}{a}\right]$$

$$=\frac{-1}{a^2}\left[\log(\sec ax+\tan ax)-\sin ax\right]$$

$$Q=\int\frac{f_1X}{f_1f_2'-f_1'f_2}dx=\int\frac{\cos ax\tan ax}{a}dx=\frac{1}{a}\int\cos ax\,\frac{\sin ax}{\cos ax}dx$$

$$=\frac{1}{a}\int\sin ax\,dx=\frac{1}{a}\left(\frac{-\cos ax}{a}\right)=\frac{-1}{a^2}\cos ax$$

$$\therefore P.I=Pf_1+Qf_2=-\frac{1}{a^2}\cos ax\left[\log(\sec ax+\tan ax)-\sin ax\right]-\frac{1}{a^2}\sin ax\cos ax$$

$$=\frac{-1}{a^2}\cos ax\log(\sec ax+\tan ax)+\frac{1}{a^2}\cos ax\sin ax-\frac{1}{a^2}\sin ax\cos ax$$

$$=\frac{-1}{a^2}\cos ax\log(\sec ax+\tan ax)$$

$\therefore y=C.F+P.I$

$$y=A\cos ax+B\sin ax-\frac{1}{a^2}\cos ax\log(\sec ax+\tan ax)$$

AU (21) Solve $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x$ by using the method of variation of parameters.

Sol: Given $(D^2+1)y=\operatorname{cosec}x$. Here $X=\operatorname{cosec}x$

Auxiliary equation is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m=\sqrt{-1}=\pm i \quad (\alpha=0, \beta=1)$

$\therefore$ C.F. is $A\cos x+B\sin x$.

Here $f_1=\cos x$, $\quad f_2=\sin x$

$f_1'=-\sin x$, $\quad f_2'=\cos x$

$$f_1f_2'-f_1'f_2=\cos x(\cos x)-(-\sin x)\sin x=\cos^2x+\sin^2x=1$$

$$P=-\int \frac{f_2X}{f_1f_2'-f_1'f_2}\,dx=-\int\frac{\sin x\operatorname{cosec}x}{1}\,dx=-\int \sin x\frac{1}{\sin x}\,dx=-\int dx$$

$$P=-x$$

$$Q=\int\frac{f_1X}{f_1f_2'-f_1'f_2}\,dx=\int\frac{\cos x\operatorname{cosec}x}{1}\,dx=\int\cos x\frac{1}{\sin x}\,dx=\int\frac{\cos x}{\sin x}\,dx$$

$$=\log(\sin x)$$

$\therefore$ P.I $=Pf_1+Qf_2=-x\cos x+\sin x\log(\sin x)$

$\therefore y=$ C.F $+$ P.I

$y=A\cos x+B\sin x-x\cos x+\sin x\log(\sin x)$.

AU (22) Solve $y''-4y'+4y=(x+1)e^{2x}$ by the method of variation of parameters.

Sol: Given $(D^2-4D+4)y=(x+1)e^{2x}$. Here $X=(x+1)e^{2x}$

$\times$	$+$
4	-4
-2	-2
$m-2$	$m-2$

Auxiliary equation is $m^2-4m+4=0$

$$(m-2)(m-2)=0$$

$$m=2,2$$

$\therefore$ C.F. is $Ae^{2x}+xBe^{2x}=Ae^{2x}+Bxe^{2x}$.

Here $f_1=e^{2x}$, $\quad f_2=xe^{2x}$

$f_1'=2e^{2x}$, $\quad f_2'=x(2e^{2x})+e^{2x}=2xe^{2x}+e^{2x}$

$$f_1f_2'-f_1'f_2=e^{2x}(2xe^{2x}+e^{2x})-2e^{2x}(xe^{2x})$$

$$=2xe^{4x}+e^{4x}-2xe^{4x}=e^{4x}$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx = -\int \frac{x e^{2x}\,(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= -\int \frac{x(x+1)e^{4x}}{e^{4x}}\,dx = -\int x(x+1)\,dx = -\int (x^2+x)\,dx$$

$$P = -\left(\frac{x^3}{3} + \frac{x^2}{2}\right)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx = \int \frac{e^{2x}(x+1)e^{2x}}{e^{4x}}\,dx = \int \frac{(x+1)e^{4x}}{e^{4x}}\,dx$$

$$= \int (x+1)\,dx = \frac{x^2}{2} + x$$

$$\therefore P.I = Pf_1 + Qf_2 = -e^{2x}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) + xe^{2x}\left(\frac{x^2}{2} + x\right)$$

$$= e^{2x}\left(-\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^3}{2} + x^2\right) = e^{2x}\left(\frac{x^3}{6} + \frac{x^2}{2}\right)$$

$$= \frac{e^{2x}}{6}\left(x^3 + 3x^2\right)$$

$\therefore y = C.F + P.I$

$$y = Ae^{2x} + Bxe^{2x} + \frac{e^{2x}}{6}(x^3+3x^2)$$

AU

(23) Solve $\frac{d^2y}{dx^2} + y = \cot x$ by using method of variation of parameters.

Sol: Given $(D^2+1)y = \cot x$. Here $X = \cot x$

Auxiliary equation is $m^2+1=0 \Rightarrow m^2=-1 \Rightarrow m = \sqrt{-1} = \pm i$ $(\alpha=0, \beta=1)$

$\therefore$ C.F. is $A\cos x + B\sin x$.

Here $f_1 = \cos x$, $f_2 = \sin x$

$f_1' = -\sin x$, $f_2' = \cos x$

$$f_1 f_2' - f_1' f_2 = \cos x(\cos x) - (-\sin x)\sin x = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx = -\int \frac{\sin x \cot x}{1}\,dx = -\int \sin x \frac{\cos x}{\sin x}\,dx$$

$$= -\int \cos x\,dx = -\sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx = \int \frac{\cos x \cot x}{1}\,dx = \int \cos x \frac{\cos x}{\sin x}\,dx = \int \frac{\cos^2 x}{\sin x}\,dx$$

$$= \int \frac{1-\sin^2 x}{\sin x}\,dx = \int (\operatorname{cosec} x - \sin x)\,dx = \log(\operatorname{cosec} x - \cot x) + \cos x$$

$\therefore P.I = Pf_1 + Qf_2 = -\sin x\cos x + \sin x\log(\operatorname{cosec}x - \cot x) + \sin x\cos x$

$= \sin x\log(\operatorname{cosec}x - \cot x)$

$\therefore y = C.F + P.I = A\cos x + B\sin x + \sin x\log(\operatorname{cosec}x - \cot x)$.

Method of undetermined coefficients:

Function of x	Choice of P.I.
① ke^{px}	① Ce^{px}
② $k\sin(ax+b)$ (or) $k\cos(ax+b)$	② $c_1\sin(ax+b)+c_2\cos(ax+b)$
③ $ke^{px}\sin(ax+b)$ (or) $ke^{px}\cos(ax+b)$	③ $c_1e^{px}\sin(ax+b)$ $+c_2e^{px}\cos(ax+b)$
④ kx^m where $m=0,1,2,\dots$	④ $C_0+C_1x+C_2x^2+\dots+C_mx^m$

AU

(24) Solve $(D^2+2D+1)y=e^x\sin 2x$ by using the method of undetermined coefficients.

×	+
1	2
1	1
$m+1$	$m+1$

Sol: Given $y''+2y'+y=e^x\sin 2x$ — ①

Auxiliary equation is $m^2+2m+1=0$

$(m+1)(m+1)=0$

$\therefore m=-1,-1$

$\therefore$ C.F. is $Ae^{-x}+Bxe^{-x}$

Solution set $S=\{e^{-x}, xe^{-x}\}$

R.H.S of the given equation is not a member of S.

Choose $P.I = y_p = c_1e^x\sin 2x + c_2e^x\cos 2x$

$y_p' = c_1\left[e^x 2\cos 2x + \sin 2x\, e^x\right] + c_2\left[e^x(-2\sin 2x) + \cos 2x\, e^x\right]$

$= 2c_1e^x\cos 2x + c_1e^x\sin 2x - 2c_2e^x\sin 2x + c_2e^x\cos 2x$

$= e^x\cos 2x(2c_1+c_2) + e^x\sin 2x(c_1-2c_2)$

$y_p' = (2c_1+c_2)e^x\cos 2x + (c_1-2c_2)e^x\sin 2x$

$y_p'' = (2c_1+c_2)\left[e^x(-2\sin 2x) + \cos 2x\, e^x\right] + (c_1-2c_2)\left[e^x 2\cos 2x + \sin 2x\, e^x\right]$

$= -2(2c_1+c_2)e^x\sin 2x + (2c_1+c_2)e^x\cos 2x + (2c_1-4c_2)e^x\cos 2x + (c_1-2c_2)e^x\sin 2x$

$= e^x\sin 2x(-4c_1-2c_2+c_1-2c_2) + e^x\cos 2x(2c_1+c_2+2c_1-4c_2)$

$y_p'' = e^x\sin 2x(-3c_1-4c_2) + e^x\cos 2x(4c_1-3c_2)$

$\therefore ① \Rightarrow (-3c_1-4c_2)e^x\sin2x+(4c_1-3c_2)e^x\cos2x+2\left[(2c_1+c_2)e^x\cos2x + (c_1-2c_2)e^x\sin2x\right]$

$+c_1e^x\sin2x+c_2e^x\cos2x = e^x\sin2x$

$\Rightarrow (-3c_1-4c_2+2c_1-4c_2+c_1)e^x\sin2x+(4c_1-3c_2+4c_1+2c_2+c_2)e^x\cos2x = e^x\sin2x$

$\Rightarrow (-8c_2)e^x\sin2x+(8c_1)e^x\cos2x = e^x\sin2x$

Equating like coefficients on both sides,

$-8c_2=1 \Rightarrow \boxed{c_2=\frac{-1}{8}}$

$8c_1=0 \Rightarrow \boxed{c_1=0}$

$\therefore P.I = y_p = \frac{-1}{8}e^x\cos2x$

$\therefore y = C.F + P.I$

$y = Ae^{-x}+Bxe^{-x}-\frac{1}{8}e^x\cos2x.$

(AU) (25) Solve $(D^2-2D)y = 5e^x\cos x$ by using method of undetermined coefficients.

Sol: Given $y''-2y' = 5e^x\cos x$ — ①

Auxiliary equation is $m^2-2m=0$

$m(m-2)=0 \Rightarrow m=0,2$

$\therefore$ C.F. is $Ae^{0x}+Be^{2x} = A+Be^{2x}$

Solution set $S=\{e^{2x}\}$.

R.H.S of equation ① is not a member of S.

Choose $P.I = y_p = c_1e^x\cos x + c_2e^x\sin x$.

$y_p' = c_1\left[e^x(-\sin x)+\cos x e^x\right]+c_2\left[e^x\cos x+\sin x e^x\right]$

$= -c_1e^x\sin x + c_1e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$

$= (-c_1+c_2)e^x\sin x + (c_1+c_2)e^x\cos x$

$y_p'' = (-c_1+c_2)\left[e^x\cos x+\sin x e^x\right]+(c_1+c_2)\left[e^x(-\sin x)+\cos x e^x\right]$

$= e^x\cos x(-c_1+c_2+c_1+c_2)+e^x\sin x(-c_1+c_2-c_1-c_2)$

$= 2c_2e^x\cos x - 2c_1e^x\sin x$

$\therefore ① \Rightarrow 2c_2 e^x \cos x - 2c_1 e^x \sin x - 2\left[(-c_1+c_2)e^x \sin x + (c_1+c_2)e^x \cos x\right] = 5e^x \cos x$

$\Rightarrow e^x \cos x(2c_2 - 2c_1 - 2c_2) + e^x \sin x(-2c_1 + 2c_1 - 2c_2) = 5e^x \cos x$

$\Rightarrow -2c_1 e^x \cos x - 2c_2 e^x \sin x = 5e^x \cos x$

Equating like coefficients on both sides,

$-2c_1 = 5 \Rightarrow \boxed{c_1 = -5/2}$

$-2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$\therefore P.I = y_p = -\frac{5}{2} e^x \cos x$

$\therefore y = C.F + P.I$

$y = A + Be^{2x} - \frac{5}{2} e^x \cos x$

AU

26 Solve $(D^2+3D+2)y = 4e^{2x} + x$ by using method of undetermined coefficients.

$\times$	$+$
2	3
1	2
$m+1$	$m+2$

Sol: Given $y'' + 3y' + 2y = 4e^{2x} + x$ — ①

Auxiliary equation is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$\therefore$ C.F. is $Ae^{-x} + Be^{-2x}$.

Solution set $S = \{e^{-x}, e^{-2x}\}$.

R.H.S. of equation ① is not a member of S.

Choose $P.I = y_p = c_1 e^{2x} + c_2 + c_3 x$

$y_p' = c_1(2e^{2x}) + c_3 = 2c_1 e^{2x} + c_3$

$y_p'' = 4c_1 e^{2x}$

$\therefore ① \Rightarrow 4c_1 e^{2x} + 3(2c_1 e^{2x} + c_3) + 2(c_1 e^{2x} + c_2 + c_3 x) = 4e^{2x} + x$

$\Rightarrow e^{2x}(4c_1 + 6c_1 + 2c_1) + 3c_3 + 2c_2 + 2c_3 x = 4e^{2x} + x$

$\Rightarrow 12c_1 e^{2x} + 2c_3 x + 3c_3 + 2c_2 = 4e^{2x} + x$

Equating like coefficients on both sides,

$12c_1 = 4 \Rightarrow c_1 = \frac{4}{12} = \frac{1}{3} \Rightarrow \boxed{c_1 = 1/3}$

$2c_3 = 1 \Rightarrow \boxed{c_3 = 1/2}$

$3c_3 + 2c_2 = 0 \Rightarrow 3(1/2) + 2c_2 = 0 \Rightarrow 2c_2 = -3/2 \Rightarrow \boxed{c_2 = -3/4}$

$\therefore P.I = y_p = \frac{1}{3}e^{2x} - \frac{3}{4} + \frac{1}{2}x$

$\therefore y = C.F + P.I$

$y = Ae^{-x} + Be^{-2x} + \frac{1}{3}e^{2x} - \frac{3}{4} + \frac{1}{2}x.$

(27) Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x} + \sin x$ by using method of underlermined coefficients.

Sol: Given $y'' - 5y' + 6y = e^{3x} + \sin x$ — ① $\Rightarrow (D^2 - 5D + 6)y = e^{3x} + \sin x$

Auxiliary equation is

×	+
6	−5
−3	−2
m−3	m−2

$m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$\therefore m = 2, 3$

$\therefore$ C.F. is $Ae^{2x} + Be^{3x}$

Solution set $S = \{e^{2x}, e^{3x}\}$.

R.H.S. of equation ① is a member of S.

Choose $P.I = c_1 x e^{3x} + c_2 \sin x + c_3 \cos x = y_p$

$y_p' = c_1[x\,3e^{3x} + e^{3x}] + c_2 \cos x - c_3 \sin x$

$= 3c_1 x e^{3x} + c_1 e^{3x} + c_2 \cos x - c_3 \sin x$

$y_p'' = 3c_1[x.3e^{3x} + e^{3x}] + 3c_1 e^{3x} - c_2 \sin x - c_3 \cos x$

$= 3c_1 e^{3x} + 9c_1 x e^{3x} + 3c_1 e^{3x} - c_2 \sin x - c_3 \cos x$

$= 6c_1 e^{3x} + 9c_1 x e^{3x} - c_2 \sin x - c_3 \cos x$

$\therefore ① \Rightarrow 6c_1 e^{3x} + 9c_1 x e^{3x} - c_2 \sin x - c_3 \cos x - 5[3c_1 x e^{3x} + c_1 e^{3x} + c_2 \cos x - c_3 \sin x]$

$+ 6[c_1 x e^{3x} + c_2 \sin x + c_3 \cos x] = e^{3x} + \sin x$

$\Rightarrow e^{3x}[6c_1 - 5c_1] + xe^{3x}[9c_1 - 15c_1 + 6c_1] + \sin x[-c_2 + 5c_3 + 6c_2]$

$+ \cos x[-c_3 - 5c_2 + 6c_3] = e^{3x} + \sin x$

$\Rightarrow c_1 e^{3x} + (5c_2 + 5c_3)\sin x + (5c_3 - 5c_2)\cos x = e^{3x} + \sin x$

Equating like coefficients on both sides,

$\boxed{c_1 = 1}$, $5c_2 + 5c_3 = 1 \Rightarrow c_2 + c_3 = 1/5 \Rightarrow 2c_2 = 1/5 \Rightarrow \boxed{c_2 = 1/10}$

$5c_3 - 5c_2 = 0 \Rightarrow 5c_3 = 5c_2 \Rightarrow \boxed{c_3 = c_2}$ $\quad\therefore \boxed{c_3 = 1/10}$

$\therefore P.I = y_p = xe^{3x} + \frac{1}{10}\sin x + \frac{1}{10}\cos x$

$\therefore y = C.F + P.I$

$y = Ae^{2x} + Be^{3x} + xe^{3x} + \frac{1}{10}\sin x + \frac{1}{10}\cos x.$

HW

(1) Solve: $(D^2+2D+2)y = e^{-2x} + \cos 2x.$

(2) Solve: $(D^3-D)y = e^x x.$

(3) Solve $(D^2+a^2)y = \sec ax$ by using method of variation of parameters.

(4) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \cot x$ by using method of variation of parameters.

(5) Solve $\frac{d^2y}{dx^2} + y = x\sin x$ by using method of variation of parameters.

(6) Solve: $x^2y'' - 4xy' + 6y = x^2 + \log x.$

(7) Solve: $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$ [Hint: $2x+3 = e^z$, $(2x+3)D = 2D'$
$(2x+3)^2D^2 = 2^2D'(D'-1)$]

(8) Solve: $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$, $\frac{dy}{dt} + 3x + 2y = 0.$

(9) Solve $(D^2-3D+2)y = 6e^{3x}$ by using method of undetermined coefficients.

(10) Solve $\frac{d^2y}{dx^2} + 9y = \cos 3x$ by using method of undetermined coefficients.